A TWO-GENERATOR PRESENTATION FOR THE PICARD GROUP

A. M. BRUNNER

(Communicated by Warren J. Wong)

ABSTRACT. The following 2-generator 5-relator presentation is given for the Picard group $\mathrm{PSL}(2, Z(i))$:

$$\begin{aligned} \langle a, w, b; b = aw^2a^{-1}w^{-2}aw^2, \\ (a^2waw^{-1})^2 = (awaw^{-1})^3 = (wb)^2 = (ab)^2 = b^2 = 1 \rangle. \end{aligned}$$

The Picard group, $P = \mathrm{PSL}(2, Z(i))$, has a well-known presentation $P = \langle u, x, y, v; u^2 = x^3 = y^3 = v^2 = (xv)^2 = (xy)^2 = (uy)^2 = (uv)^2 = 1 \rangle$ ([2] or [1]). This presentation comes from looking at rotations on the edges of a fundamental polyhedron for P in $\mathbb{H}^3$. With the usual convention for matrices in PSL , a faithful representation is given with

$$u = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad x = \begin{bmatrix} 0 & i \\ i & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad v = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We show below that P is generated by

$$xu = \begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix} \quad \text{and} \quad vuy = \begin{bmatrix} i & i \\ i & 0 \end{bmatrix}$$

in a presentation with five relators. The reduction to three generators is standard for presentations of this sort; the reduction to two generators is more unexpected.

The first reduction in generators is obtained with $a = xu$, $b = uy$ so that $x = au$, $y = ub$. Thus

$$P = \langle u, a, b, v; u^2 = (au)^3 = (ub)^3 = v^2 = (auv)^2 = (ab)^2 = b^2 = (uv)^2 = 1 \rangle.$$

Now, using $(uv)^2 = (au)^3 = u^2 = v^2 = 1$, we have $auvauv = avuauv = avau^{-1}u^{-1}a^{-1}v$. It follows that we may replace the relation $(auv)^2 = 1$ with $u = a^{-1}vava^{-1}$. Consequently,

$$\begin{aligned} P = \langle a, b, v; (a^{-1}vava^{-1})^2 = (vava^{-1})^3 = (a^{-1}vava^{-1}b)^3 \\ = v^2 = (ab)^2 = b^2 = (a^{-1}vava^{-1}v)^2 = 1 \rangle. \end{aligned}$$

Moreover, the relation $(a^{-1}vava^{-1}v)^2 = 1$ can be omitted, since from $(a^{-1}vava^{-1})^2 = v^2 = 1$, $1 = vava^{-1}vava^{-1}vava^{-1} = vavava^{-1}vavava^{-1} = (vavava^{-1})^2 = (ava^{-1}va^{-1}v)^{-2}$.

Received by the editors October 25, 1990 and, in revised form, November 30, 1990.
1980 *Mathematics Subject Classification* (1985 Revision). Primary 20F05.

For the second reduction, let $w = vb$, so that $v = wb$. Then

$$\begin{aligned} P &= \langle a, b, w; (a^{-1}wbawba^{-1})^2 = (wbawba^{-1})^3 \\ &= (a^{-1}wbawba^{-1}b)^3 = (wb)^2 = (ab)^2 = b^2 = 1 \rangle. \end{aligned}$$

Here $b^{-1}ab = a^{-1}$, $b^{-1}wb = w^{-1}$, $b^2 = 1$. The relation $(a^{-1}wbawba^{-1}b)^3 = 1$ can be replaced by $(a^{-1}wbawba)^3 = 1$ or $(w^2ba)^3 = 1$. From this last relation, $1 = w^2baw^2baw^2ba = w^2a^{-1}w^{-2}aw^2ba$ so that $b = aw^2a^{-1}w^{-2}aw^2$. Thus P is generated by $a = xu$ and $w = vuy$ as claimed.

Simplifying a little, we obtain

$$\begin{aligned} P &= \langle a, w, b; b = aw^2a^{-1}w^{-2}aw^2, \\ &= (a^2waw^{-1})^2 = (awaw^{-1})^3 = (wb)^2 = (ab)^2 = b^2 = 1 \rangle. \end{aligned}$$

It is interesting to observe the central role played by the unimodular group $\text{sgp}\{a, waw^{-1}\}$. We have not succeeded in finding a more economical presentation for P than this.

REFERENCES

1. A. M. Brunner, M. L. Frame, Y. W. Lee, and N. J. Wielenberg, *Classifying torsion-free subgroups of the Picard group*, Trans. Amer. Math. Soc. **282** (1984), 205–235.
2. Benjamin Fine, *The HNN and generalized free product structure of certain linear groups*, Bull. Amer. Math. Soc. **81** (1975), 413–416.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-PARKSIDE, KENOSHA, WISCONSIN 53141