SHADOWING PROPERTY OF CONTINUOUS MAPS

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ABSTRACT. We study continuous maps of an interval into itself. We find the necessary and sufficient condition for the maps of the type 2^n to have the shadowing property. Further we show that any chaotic map, which has only cycles of order a power of 2, does not have the shadowing property.

INTRODUCTION

Let $f: I(= \langle 0; 1 \rangle) \rightarrow I$ be a continuous map of the interval I to itself (i.e., $f \in C^0(I, I)$). The orbit of x is the set $\{x, f(x), f^2(x), \dots\}$ and let $f_\delta(x) := \langle f(x) - \delta; f(x) + \delta \rangle$. Given $\delta > 0$, a δ -chain of x (or δ -pseudo orbit of x) is a sequence $X_\delta = \{x_n\}_{n=0}^\infty$, where $x_{i+1} \in f_\delta(x_i)$ and $x_0 = x$.

The investigation of the pseudo orbits is very important in connection with the calculation of the orbits by a computer, because a computer can calculate only pseudo orbits. For example, if a computer has accuracy in computation of $f(x)$ to 20 decimal places, we can regard x_{i+1} as the truncation of $f(x_i)$ to 20 decimal places so that $|x_{i+1} - f(x_i)| < 2^{-20}$. Therefore it is very interesting to know when the pseudo orbits can be approximated by actual orbits.

We say that δ -chain X_δ is ε -shadowed by an orbit of y if for every n

$$|x_n - f^n(y)| < \varepsilon.$$

Now, if for every $\varepsilon > 0$ there is such a $\delta > 0$ that every X_δ is ε -shadowed by some $y \in I$, then f has the shadowing property.

The main aim of this paper is to find the class of continuous maps with the shadowing property. We will assume that $\text{Per}(f) = \text{Fix}(f)$ and that $\text{Fix}(f)$ is nowhere dense in I . While the first assumption is technical and we can easily extend our result to the maps with $\text{Per}(f^n) = \text{Fix}(f^n)$, i.e., on maps of the type 2^n (see Theorem 2), the second assumption is a necessary one for the shadowing property. We abandoned the very restrictive assumption, that f is a homeomorphism, which was used in [3].

In the paper in [1] the family of tent maps was investigated and the residual

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set of parameters s was found, for which f_s has the shadowing property.

PRELIMINARIES

We will denote a closed interval with $x \leq y$ by $\langle x; y \rangle$ and a closed interval where no information about order of x, y is provided, by $\langle x; y \rangle^*$.

In the sequel, we will use the following lemmas of Šarkovskii [4]:

Lemma 1 (Šarkovskii [4]). *Let $f \in C^0(I; I)$ and $\text{Per}(f) = \text{Fix}(f)$. Then for any $x \in I$, if $f(x) > x$ ($f(x) < x$), then for all $y \in \langle x; f(x) \rangle$ (respectively $y \in \langle f(x); x \rangle$) we have $f(y) > x$ (respectively $f(y) < x$).*

Lemma 2 (Šarkovskii [4]). *Let $f \in C^0(I; I)$ and $\text{Per}(f) = \text{Fix}(f)$. Then for all $n_1, n_2, n_3 \in N$; $n_1 < n_2 < n_3$, and for all $x \in I$, if $f^{n_2}(x) \neq f^{n_3}(x)$, then $f^{n_1}(x) \notin \langle f^{n_2}(x); f^{n_3}(x) \rangle^*$.*

Lemma 3 (Šarkovskii [4]). *Let $f \in C^0(I; I)$ and $\text{Per}(f) = \text{Fix}(f)$. Then for any $x \in I$, $\lim_{n \rightarrow \infty} f^n(x) = p$, where $p \in \text{Fix}(f)$.*

Before stating the main result we prove several lemmas.

Lemma 4. *Let $f \in C^0(I; I)$ and $\text{Per}(f) = \text{Fix}(f)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1, x_2, x_3 \in I$ with $x_1 \in \langle x_3; x_2 \rangle^*$, and $x_2 \in f_\delta(x_1)$, and $x_3 \in f_\delta(x_2)$, we have $|x_2 - x_3|_* < \varepsilon$. (It is an analogue of Lemma 2.)*

Proof. Let

- (1) $\varepsilon_\delta^1 = \sup\{|f(x) - f(y)|; |x - y| \leq \delta\}$
- (2) $\varepsilon_\delta^2 = \sup\{|x - y|; x \in f_\delta(y) \text{ and } y \in f_\delta(x)\}$
- (3) Clearly, if $\delta \rightarrow 0$ then $\varepsilon_\delta^1, \varepsilon_\delta^2 \rightarrow 0$.

Without loss of generality assume that $x_3 \leq x_1 \leq x_2$. We will discuss several cases.

- (A) Let $f(x_1) \leq x_1$. Then $|x_2 - x_1| \leq \delta$ and by (1) we have that $|f(x_2) - f(x_1)| \leq \varepsilon_\delta^1$. Hence $|x_3 - x_2| \leq \varepsilon_\delta^1 + 2\delta$.
- (B1) Let $f(x_1) > x_1$ and $f(x_2) \geq x_1$. Since $x_3 \leq x_1$ then $x_1 \in f_\delta(x_2)$ and by (2) we have $|x_1 - x_2| \leq \varepsilon_\delta^2$. Finally since $f(x_2) \geq x_1 \geq x_3$ we have $|x_3 - x_2| \leq \varepsilon_\delta^2 + \delta$.
- (B2) Let $f(x_1) > x_1$ and $f(x_2) < x_1$. From Lemma 1 we have that $x_2 > f(x_1)$ and so there exists $y \in \langle x_1; x_2 \rangle$ such that $f(y) = x_1$. Again using Lemma 1 we have $y \in \langle f(x_1); x_2 \rangle$, then $y \in f_\delta(x_1)$ and (2) implies $|x_1 - y| \leq \varepsilon_\delta^2$. Hence $|x_1 - x_2| \leq \varepsilon_\delta^2 + \delta$.

Finally, since $|y - x_2| \leq \delta$ by (1) we have $|x_1 - f(x_2)| \leq \varepsilon_\delta^1$ and $|x_1 - x_3| \leq \varepsilon_\delta^1 + \delta$.

Now clearly $|x_2 - x_3| \leq \varepsilon_\delta^1 + \varepsilon_\delta^2 + 2\delta$. Hence for x_1, x_2, x_3 satisfying assumptions of our lemma we have that $|x_2 - x_3| \leq \varepsilon_\delta^1 + \varepsilon_\delta^2 + 2\delta$ and (3) finishes the proof. $\square$

Lemma 5. *Let $f \in C^0(I; I)$ and $\text{Per}(f) = \text{Fix}(f)$. Let $0 = d_0 < d_1 < \dots < d_n = 1$ be a division of I such that $d_i \notin \text{Fix}(f)$ for $i \in \{1, \dots, n-1\}$. Denote by $I_j = \langle d_{j-1}; d_j \rangle$ for $j \in \{1, \dots, n\}$. Then there exists such a $\delta > 0$*

that if for some $m \in \mathbb{N}$ there are points $x_1, \dots, x_m$ with $x_{i+1} \in f_\delta(x_i)$ for $i \in \{1, \dots, m-1\}$ and $x_1 \in f_\delta(x_m)$ (i.e., $x_1, \dots, x_m$ are δ - m -cycle), then there is a j such that $x_i \in I_j$ for every i (i.e., all δ -cycles for sufficiently small δ are "inside" the intervals I_j).

Proof. Let $2\varepsilon = \min\{|d_i - p|; p \in \text{Fix}(f) \text{ and } i \in \{1, \dots, n-1\}\}$. ($\text{Fix}(f)$ is closed set and $d_i \notin \text{Fix}(f)$.) For this ε we can find a $\delta > 0$ for which Lemma 4 is satisfied and

$$(4) \quad x \notin f_\delta(x) \quad \text{for every } x \in \langle d_i - \varepsilon; d_i + \varepsilon \rangle$$

(because f is uniformly continuous).

We claim that this δ satisfies Lemma 5. To the contrary suppose that there exists an δ -cycle $x_1, \dots, x_m$ and $d := d_i$ such that $d \in (x_j; x_k)^*$ for some j and k .

Without loss of generality assume that $f(d) > d$, $x_1 \leq d$, and $x_2 > d$ and x_1 is maximal point with this property, i.e.

$$(5) \quad \text{if } x_i \leq d \text{ and } x_{i+1} > d \text{ then } x_i \leq x_1.$$

Let $x_j \geq d$, $x_{j+1} < d$, and let j be minimal index with this property, i.e.

$$(6) \quad x_i \geq d \quad \text{for } i \in \{2, \dots, j\}.$$

Clearly $x_j > d + \varepsilon$. We will again discuss two cases.

- (A) Suppose $x_{j+1} \leq x_1$. Then there exists $k \in \{1, \dots, j-1\}$ such that $x_1 \leq x_k < x_j \leq x_{k+1}$ (see (6)) and there is a $y \in \langle x_k; x_j \rangle$ such that $x_j \in f_\delta(y)$ (it follows from the continuity of the relation f_δ). So we have $y \in \langle x_{j+1}; x_j \rangle$, $x_j \in f_\delta(y)$, and $|x_j - x_{j+1}| > \varepsilon$, hence a contradiction with Lemma 4.
- (B) Now suppose $x_{j+1} > x_1$. Then we have $x_{j+1} < d - \varepsilon$ (see (4) and (5)) and then also $x_1 < d - \varepsilon$. Now from (5) similarly as in (A) we obtain that there is a $y \in \langle x_1; d - \varepsilon \rangle$ with $x_1 \in f_\delta(y)$. So we have $y \in \langle x_1; x_2 \rangle$, $x_1 \in f_\delta(y)$, and $|x_2 - x_1| > \varepsilon$ —again a contradiction with Lemma 4. $\square$

Definition 1. Let us consider $\mathbf{X}_\delta = \{x_i\}_{i=0}^\infty$ and define numbers $K^k = \sum_{i=0}^\infty \alpha_{ik}$ where

$$\begin{aligned} \alpha_{ik} &= 1 \text{ if there is such } j \in \{0, \dots, i\} \text{ that } x_{i-j} \in \text{int}(I_k), x_{i+1} \notin I_k, \\ &\text{and } x_{i-j+1} \in I_k \text{ for } 1 \leq l \leq j \\ \alpha_{ik} &= 0 \text{ otherwise} \end{aligned}$$

(i.e., K^k is the number of departures from I_k).

We will call $K = \sum_{k=1}^n K^k$ the interval index of $\mathbf{X}_\delta$.

Lemma 6. Let $f \in C^0(I; I)$ and $\text{Per}(f) = \text{Fix}(f)$. Let us consider the same division $\{d_i\}_{i=0}^n$ as in Lemma 5. Then there exists $\delta > 0$ and $N_0 \in \mathbb{N}$ such that the interval index K of every $\mathbf{X}_\delta$ is less than N_0 .

Proof. Take a δ such that all δ -cycles are "inside" the intervals I_j (see Lemma 5). Assume to the contrary that our statement is not true. Then there exists such an $\mathbf{X}_\delta$ for which $K > nm$ where n is the number of intervals I_j and $m > \max_{j \in \{1, \dots, n\}} \{2|I_j|/\delta\}$.

Then there exists an interval I_α for which $K^\alpha > m$. Indeed, let I_α be an interval for which $K^\alpha = \max_{j \in \{1, \dots, n\}} \{K^j\}$. We have $K^\alpha \geq K/n > m$.

If for some i there is such $j \in \{0, \dots, i\}$ that $\mathbf{x}_{i-j} \in \text{int}(I_\alpha)$, $\mathbf{x}_{i+1} \notin I_\alpha$, and $\mathbf{x}_{i-j+l} \in I_\alpha$ for $1 \leq l \leq j$, then we will denote such an $\mathbf{x}_i$ as $\hat{\mathbf{x}}_i$.

But because $K^\alpha > m$ then we obtain that there are more than m such a points $\hat{\mathbf{x}}_i$ and by the Dirichlet principle we can find such $\hat{\mathbf{x}}_i, \hat{\mathbf{x}}_j \in I_\alpha$; $i < j$, that $|\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j| < \delta/2$ and so $|f(\mathbf{x}_{j-1}) - \hat{\mathbf{x}}_i| \leq \delta$. Hence $\{\hat{\mathbf{x}}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_{j-2}, \mathbf{x}_{j-1}\}$ is a δ -cycle, which is not in a single interval I^α . Hence we have a contradiction to our assumption that all δ -cycles are "inside" the intervals I_j . $\square$

Definition 2. We will call a one-side neighborhood $\langle p; q \rangle^*$ of the fixed point p a nontrapping neighborhood of p if for every $x \in \langle p; q \rangle^*$ we have $x \in f(\langle p; x \rangle^*)$.

Lemma 7. Let $f \in C^0(I; I)$ and $\text{Per}(f) = \text{Fix}(f)$. Let $x \in I$ be such that $f^n(x) \rightarrow p$, $p \in \text{Fix}(f)$, and let $\langle p; q \rangle^*$ be a nontrapping neighborhood of p . Then if $f^n(x) \in \langle p; q \rangle^*$ for some $n \in N$, then we have $f^n(x) = p$.

Proof. Let $f^n(x) \in \langle p; q \rangle^*$. Because $\langle p; q \rangle^*$ is a nontrapping neighborhood of p there is a $z \in \langle p; f^n(x) \rangle^*$ such that $f(z) = f^n(x)$. But because $f^i(x) \rightarrow p$ we obtain from Lemma 2 that $z = p$. Hence $f^n(x) = p$. $\square$

MAIN RESULTS

The following lemma is very technical and is needed in the proof of the Main Theorem.

Lemma 8. Let $f \in C^0(I; I)$ and $\text{Per}(f) = \text{Fix}(f)$. Let $0 = d_0 < d_1 < \dots < d_n = 1$ be the division of I as in Lemma 5. Let us consider some sequence of δ -chains $\{\mathbf{X}_{\delta_j}\}_{j=0}^\infty = \{\{\mathbf{x}_i^j\}_{i=0}^\infty\}_{j=0}^\infty$ where $\lim_{j \rightarrow \infty} \delta_j = 0$ and $\lim_{j \rightarrow \infty} \mathbf{x}_0^j = x$.

Let $\lim_{n \rightarrow \infty} f^n(x) = p$, $p \in I_k$, and let there exist such $n_0 \in N$ such that if $n \geq n_0$, then $f^n(x) \in \text{int}(I_k)$.

Suppose that there is such a $j_0 \in N$ that for every $j \geq j_0$ there is such $i(j) \geq n_0$ that for $n_0 \leq i \leq i(j)$

$$(7) \quad \mathbf{x}_{n_0}^j \in \text{int}(I_k) \text{ and } \mathbf{x}_i^j \in I_k \text{ and } \mathbf{x}_{i(j)+1}^j \notin I_k.$$

Let $\{\mathbf{x}_{i(j)}^j\}_{j=j_0}^\infty$ have a limit. Set $y = \lim_{j \rightarrow \infty} \mathbf{x}_{i(j)}^j$.

If $y \geq p$, then $y > p$ and $f(y) > y$ (if $y \leq p$ then $y < p$ and $f(y) < y$) and $\langle p; f(y) \rangle^*$ is a nontrapping neighborhood of p .

Proof. Without loss of generality let $y \geq p$. Then $f(y) \notin \text{int}(I_k)$ by continuity of f and (7). Hence we have $y > p$ (because $p \in \text{Fix}(f)$ by Lemma 3).

Suppose that $f(y) < y$. Then $f(y) < p$ and since p is the limit point of $f^n(x)$ and y is the limit point of $\mathbf{x}_{i(j)}^j$, then there is such $j_1 \geq j_0$ that for all $j \geq j_1$ there is $s \in \{n_0, \dots, i(j)\}$ such that

$$(8) \quad \mathbf{x}_{i(j)+1}^j < \frac{p + f(y)}{2} < \mathbf{x}_s^j < \frac{p + y}{2} < \mathbf{x}_{i(j)}^j.$$

Then in the same way as in Lemma 5 we can find a $z_j \in \langle d_{k-1}; \mathbf{x}_{i(j)}^j \rangle$ such that $\mathbf{x}_{i(j)}^j \in f_\delta(z_j)$ (see (7)).

Hence by (8), for every $j \geq j_1$ we have

$$\mathbf{x}_{i(j)+1}^j \leq z_j \leq \mathbf{x}_{i(j)}^j \quad \text{where } |\mathbf{x}_{i(j)}^j - \mathbf{x}_{i(j)+1}^j| > \frac{y - f(y)}{2}$$

which is a contradiction to Lemma 4.

Hence we have $f(y) \geq y$ and since $d_k \notin \text{Fix}(f)$ we have $f(y) > y$.

To prove the second part of the lemma we observe that $\langle p; y \rangle$ is a nontrapping neighborhood of p .

Assume to the contrary that $\langle p; y \rangle$ is not a nontrapping neighborhood of p .

Then there is a $z \in \langle p; y \rangle$ such that $z \notin f(\langle p; z \rangle)$ and we have for all $z_1 \in f(\langle p; z \rangle)$; $z_1 < z$. If we take i sufficiently large (δ_i sufficiently small), then $\mathbf{x}_k^i \in \langle p; z \rangle$ implies $\mathbf{x}_{k+1}^i \leq z - \varepsilon_z$ where $\varepsilon_z > 0$ and $z - \varepsilon_z > p$ (clearly $z \neq p$).

Hence if $\mathbf{x}_{k+1}^i > z - \varepsilon_z$ (such an $\mathbf{x}_{k+1}^i$ exists because $y = \lim_{i \rightarrow \infty} \mathbf{x}_{i(j)}^j$) then $\mathbf{x}_k^i < p$ and by uniform continuity we can find $i_0 \in N$ and $\omega > 0$ such that $\mathbf{x}_{k+1}^i > z - \varepsilon_z$ implies $\mathbf{x}_k^i < p - \omega$ for $i \geq i_0$.

Then similarly as above we can find such $i_1 \geq i_0$ that for all $i \geq i_1$ there are such $j, k \in N$ that

$$\begin{aligned} \mathbf{x}_{j+k}^i &< p - \omega < \mathbf{x}_j^i < z - \varepsilon_z < \mathbf{x}_{j+k+1}^i \\ \mathbf{x}_{j+l}^i &< z - \varepsilon_z \quad \text{for } 0 \leq l \leq k \end{aligned}$$

and the same argument as above leads to contradiction to Lemma 4.

Hence $\langle p; y \rangle$ is a nontrapping neighborhood of p and clearly $\langle p; f(y) \rangle$ has the same property. $\square$

Main Theorem. Let $f \in C^0(I; I)$, $\text{Per}(f) = \text{Fix}(f)$, and $\text{Fix}(f)$ be nowhere dense. Then the following statements are equivalent.

- (i) If for some $x \in I$ $f^n(x)$ converges to the fixed point p and there exists a nontrapping neighborhood $\langle p; q \rangle^*$ of p , then for all O_x (neighborhoods of x) and for all $z \in \langle p; q \rangle^*$ there is an $n \in N$ such that $\langle p; z \rangle^* \subset f^n(O_x)$.
- (ii) f has the shadowing property.

Proof of $\square$. Suppose that f does not have the shadowing property. Then there exists an $\varepsilon > 0$ and a sequence of δ -chains $\{\mathbf{X}_{\delta_j}\}_{j=1}^\infty$ where $\delta_j < 1/j$ and $\mathbf{X}_{\delta_j}$ is not ε -shadowed.

Now we find the division of I as in Lemma 5 with the property

$$(9) \quad \max_{0 \leq i \leq n-1} |d_{i+1} - d_i| < \varepsilon.$$

(Here we used the assumption that $\text{Fix}(f)$ is nowhere dense.)

According to Lemma 6 there is such a $\delta > 0$ and $N_0 \in N$ that an interval index K of any $\mathbf{X}_\delta$ is less than N_0 . Without loss of generality we can assume that $\delta_j \leq \delta$ for all $j \in N$ and so

$$(10) \quad K_j < N_0 \quad \text{where } K_j \text{ is the interval index of } \mathbf{X}_{\delta_j}.$$

Put

$$(11) \quad x = \lim_{j \rightarrow \infty} \mathbf{x}_0^j$$

(if necessary we can consider the proper subsequence), i.e., x is the limit point of “starting points” of X_{δ_j} . From Lemma 3 we have

$$(12) \quad \lim_{n \rightarrow \infty} f^n(x) = p \text{ and } p \in \text{Fix}(f) \text{ and } p \in \text{int}(I_k) \text{ for some } k.$$

Since $f^n(x)$ converges to p , there exists n_0 such that

$$(13) \quad f^n(x) \in \text{int}(I_k) \text{ for all } n \geq n_0$$

and by continuity there is $j_0 \in N$ such that for all $j \geq j_0$

$$|x_i^j - f^i(x)| < \varepsilon \text{ for } 0 \leq i \leq n_0 \\ x_{n_0}^j \in \text{int}(I_k)$$

If there is such an $s \geq j_0$ that for all $i > n_0$; $x_i^s \in I_k$ then clearly X_{δ_s} is ε -shadowed by orbit of x which is in contradiction with our assumption. Hence for all $j \geq j_0$ there is $i(j) \geq n_0$ such that

$$x_{i(j)}^j \in I_k \text{ and } x_{i(j)+1}^j \notin I_k$$

and let us consider only the smallest numbers with this property. Hence for $j \geq j_0$ we have

$$(14) \quad x_{n_0}^j \in \text{int}(I_k) \text{ and } x_i^j \in I_k \text{ and } x_{i(j)+1}^j \notin I_k \text{ for } n_0 \leq i \leq i(j).$$

Set

$$(15) \quad y = \lim_{j \rightarrow \infty} x_{i(j)}^j$$

(if necessary we can consider the proper subsequence).

Without loss of generality assume $y \geq p$. Then by (11) to (15) and by Lemma 8 we have $p < y < f(y)$ and $\langle p; f(y) \rangle$ is the nontrapping neighborhood of p .

Now we can find $n_1 \geq n_0$ such that $f^n(x) \leq p$ for all $n \geq n_1$ (it is possible by Lemma 7) and by continuity there are $j_1 \geq j_0$, $\alpha > 0$ such that for all $j \geq j_1$

$$(16) \quad |x_i^j - f^i(z)| < \varepsilon \text{ for } 0 \leq i \leq n_1, \ z \in \langle x - \alpha; x + \alpha \rangle = O_\alpha$$

$$(17) \quad f^{n_1}(O_\alpha) \subset \left\langle d_{k-1}; \frac{p+y}{2} \right\rangle.$$

According to our assumption there exists an $n_\alpha \in N$ such that $\langle p; (y + f(y))/2 \rangle \subset f^{n_\alpha}(O_\alpha)$ and so there is a neighborhood O_y of y such that

$$(18) \quad O_y \subset f^{n_\alpha}(O_\alpha) \text{ and } O_y \subset \left\langle \frac{p+y}{2}; d_k \right\rangle.$$

Since $\langle p; f(y) \rangle$ is a nontrapping neighborhood of p , we obtain that $O_y \subset f^{n_\alpha+l}(O_\alpha)$ for all $l \geq 0$.

Claim. We claim that for every number $n_\alpha + l$ there is a neighborhood $O_\beta = \langle x - \beta_1; x + \beta_2 \rangle$; $O_\beta \subset O_\alpha$, such that

$$(*) \quad f^s(O_\beta) \subset I_k \text{ for } n_1 \leq s \leq n_\alpha + l$$

$$(**) \quad O_y \subset f^{n_\alpha+l}(O_\beta).$$

Proof. We have $f^{n_1}(O_\alpha) \subset I_k$ and by the continuity of f we can find such an $O_{\alpha_1} = \langle x - \alpha_1; x + \alpha_2 \rangle \subset O_\alpha$ that $f^{n_1+1}(O_{\alpha_1}) \subset I_k$ and O_{α_1} is maximal with these properties (we simply "cut off" $f^{n_1+1}(O_\alpha) \setminus I_k$) and by induction we can find $O_{\alpha_i} \subset O_{\alpha_{i-1}}$ with $f^{n_1+i}(O_{\alpha_i}) \subset I_k$ and finally for $i = n_\alpha + l - n_1$ we obtain O_β with the property (*).

If $f^{n_\alpha+l}(O_\alpha \setminus O_\beta) \cap O_y = \emptyset$ then we have $O_y \subset f^{n_\alpha+l}(O_\beta)$ (see (18)).

Suppose now that there is a $z \in f^{n_\alpha+l}(O_\alpha \setminus O_\beta) \cap O_y$. Then there exists such a $z_1 \in O_\alpha$ ($z_1 \in f^{-(n_\alpha+l)}(z)$) that

$$f^{n_1}(z_1) \in \left\langle d_{k-1}; \frac{p+y}{2} \right\rangle \text{ and } f^{n_\alpha+l}(z_1) \in \left\langle \frac{p+y}{2}; d_k \right\rangle \text{ and } f^s(z_1) \notin I_k$$

for some $n_1 < s < n_\alpha + l$ (see (17), (18)). Let s be the smallest number with this property. From Lemma 2 we obtain that $f^s(z_1) > d_k$. Because $f^s(x) \leq p$ then $\langle p; d_k \rangle \subset f^s(O_{\alpha_{s-n_1}})$ (s was chosen to be the smallest one) and since $\langle p; d_k \rangle$ is a nontrapping neighborhood of p (because $\langle p; d_k \rangle \subset \langle p; f(y) \rangle$) we obtain that $O_y \subset f^{n_\alpha+l}(O_\beta)$. $\square$

Take such an $r_1 \geq j_1$ that for all $j \geq r_1$; $i(j) > n_\alpha$. Then for any $\mathbf{X}_{\delta_j}$, $j \geq r_1$; according to (9), (14), (16) and the claim we can find an interval $R_j^1 \subset O_\alpha$ such that for all $z \in R_j^1$

$$(19) \quad O_y \subset f^{i(j)}(R_j^1) \text{ and } |f^s(z) - \mathbf{x}_s^j| < \varepsilon \text{ for } 0 \leq s \leq i(j).$$

Now we have the sequence of δ -chains $\{\mathbf{X}_{\delta_j}\}_{j \geq r_1}^\infty$ with starting points $\mathbf{x}_{i(j)}^j$ and $\lim_{j \rightarrow \infty} \mathbf{x}_{i(j)}^j = y$. Again these δ_j -chains $\{\mathbf{x}_i^j\}_{i=i(j)}^\infty$ are not ε -shadowed by any points of O_y for all $j \geq r_1$. Indeed, if there is such a $j \geq r_1$ that $\{\mathbf{x}_i^j\}_{i=i(j)}^\infty$ is ε -shadowed by some point of O_y , then from (19) we have that this $\mathbf{X}_{\delta_j}$ is ε -shadowed by some point of R_j^1 which is in contradiction with our assumption.

So we have the same situation as in the beginning, but the very important property is that $\mathbf{x}_s^j \in \text{int}(I_k)$ for some $s \in \{0, \dots, i(j)\}$ and $\mathbf{x}_{i(j)+1}^j \notin I_k$ for all $j \geq r_1$. So we have that $K_j \geq 1$ for all $j \geq r_1$.

If we repeat this procedure once more, we obtain that there are $r_2 \geq r_1$, $y_2 \in I_{k_2}$, and O_{y_2} a neighborhood of y_2 such that for all $j \geq r_2$ there are such $i(j)_2 \geq i(j)$ and $R_j^2 \subset O_y$ that for all $z \in R_j^2$

$$\begin{aligned} O_{y_2} &\subset f^{i(j)_2}(R_j^2) \\ |f^s(z) - \mathbf{x}_s^j| &< \varepsilon \text{ for } i(j) \leq s \leq i(j)_2 \\ \lim_{j \rightarrow \infty} \mathbf{x}_{i(j)_2}^j &= y_2 \\ \{\mathbf{x}_i^j\}_{i=i(j)_2}^\infty &\text{ is not } \varepsilon\text{-shadowed by any points of } O_y \end{aligned}$$

and, of course, we have

$$K_j \geq 2 \text{ for all } j \geq r_2.$$

But if we repeat this procedure N_0 times we obtain that there is such an $r_{N_0} \in \mathbb{N}$ that $K_j \geq N_0$ for all $j \geq r_{N_0}$ which is a contradiction to (10). $\square$

Proof of $\boxed{\uparrow}$. Let $x \in I$, $f^n(x) \rightarrow p$, and $\langle p; q \rangle^*$ be a nontrapping neighborhood of p and let there exist a $z \in \langle p; q \rangle^*$ and a neighborhood O_x of x such that

$$(20) \quad \langle p; z \rangle^* \not\subset f^n(O_x) \quad \text{for all } n \in \{0, 1, \dots\}.$$

Without loss of generality assume that $\langle p; q \rangle^* = \langle p; q \rangle$. Let $\varepsilon > 0$ such that

$$(21) \quad |q - z| > \varepsilon \text{ and } \langle x - \varepsilon; x + \varepsilon \rangle \subset O_x.$$

Now we claim that for any $\delta > 0$ there exists X_δ which is not ε -shadowed.

By Lemma 7 there is an $n_\delta \in N$ such that

$$(22) \quad f^n(x) \leq p \quad \text{for all } n \geq n_\delta$$

$$(23) \quad |p - f^{n_\delta}(x)| < \delta.$$

Let $k_\delta \in N$ be such that $k_\delta > |q - p|/\delta$. Set $\mathbf{x}_{n_\delta+k_\delta} = q$ and for $i \in \{1, \dots, k_\delta\}$ we define:

$$\begin{aligned} &\text{if } \mathbf{x}_{n_\delta+k_\delta-i+1} - \delta \leq p \text{ then } \mathbf{x}_{n_\delta+k_\delta-i} = p \\ &\text{if } \mathbf{x}_{n_\delta+k_\delta-i+1} - \delta > p \text{ then } f(\mathbf{x}_{n_\delta+k_\delta-i}) = \mathbf{x}_{n_\delta+k_\delta-i+1} - \delta \text{ and } \mathbf{x}_{n_\delta+k_\delta-i} \in \\ &\quad \langle p; \mathbf{x}_{n_\delta+k_\delta-i+1} - \delta \rangle \end{aligned}$$

(this definition is correct because $\langle p; q \rangle$ is a nontrapping neighborhood of p) and let $\mathbf{x}_i = f^i(x)$ for $i \in \{0, \dots, n_\delta - 1\}$.

Hence $\{\mathbf{x}_i\}_{i=0}^{n_\delta+k_\delta}$ is a δ -chain (see (23)). If $y \notin O_x$ then $|y - \mathbf{x}_0| > \varepsilon$ (see (21)) and if $y \in O_x$ then $f^{n_\delta+k_\delta}(y) < z$ (see (20) and (22)) and so $|\mathbf{x}_{n_\delta+k_\delta} - f^{n_\delta+k_\delta}(y)| > \varepsilon$ (see (21)).

So our δ -chain cannot be ε -shadowed. $\square$

Remark 1. From the Main Theorem we obtain that if f has the shadowing property, $\text{Per}(f) = \text{Fix}(f)$ and $p \in \text{Fix}(f)$, then the following condition is fulfilled:

If $\langle p; q \rangle^*$ is the maximal nontrapping neighborhood of p , then there exists such a sequence $\{x_i\}_{i=-\infty}^\infty$ that $f(x_{i-1}) = x_i$, $\lim_{i \rightarrow -\infty} x_i = p$, and

$$\inf\{|x_i - q|; i \in Z\} = 0.$$

If we consider only nondecreasing maps then this condition is equivalent to the shadowing property and thus with the result of [3]. This means that there cannot be such a fixed point p with a neighborhood O_p that for all $x \in O_p$; $f(x) \geq x$ (or for all $x \in O_p$; $f(x) \leq x$) (see Figure 1).

Lemma 9. Let $f \in C^0(I, I)$ and $n \in N$. Then f has the shadowing property if and only if f^n has the shadowing property.

Proof.

$\boxed{\Rightarrow}$ If $\{\mathbf{x}_i^n\}_{i=0}^\infty$ is a δ -chain of f^n then $\{\mathbf{x}_i^n, f(\mathbf{x}_i^n), \dots, f^{n-1}(\mathbf{x}_i^n)\}_{i=0}^\infty$ is a δ -chain of f and we are done.

$\boxed{\Leftarrow}$ From uniform continuity we obtain that

(*) for all $\varepsilon > 0$ there is such $\delta, \varepsilon_n > 0$ that if $\{\mathbf{x}_i\}_{i=0}^\infty$ is δ -chain and $\{\mathbf{x}_{ni}\}_{i=0}^\infty$ is ε_n -shadowed by f^n , then $\{\mathbf{x}_i\}_{i=0}^\infty$ is ε -shadowed by f ;

FIGURE 1

FIGURE 2

FIGURE 3

(**) for all $\delta_n > 0$ there is such $\delta \in (0; \delta_n)$ that if $\{x_i\}_{i=0}^{\infty}$ is δ -chain of f than $\{x_{ni}\}_{i=0}^{\infty}$ is δ_n -chain of f^n .

From here we are easily done. $\square$

Definition 3. We will call a one-side neighborhood $\langle p; q \rangle^*$ of the periodic point p a m -nontrapping neighborhood of p if $f^m(p) = p$ and for every $x \in \langle p; q \rangle^*$, $x \in f^m(\langle p; x \rangle^*)$.

Theorem 2. Let $f \in C^0(I; I)$ and $\text{Per}(f)$ be a closed (i.e., f is of type 2^n) nowhere dense set. Then the following statements are equivalent:

- (i) If for some $x \in I$ $f^n(x)$ converges to the periodic point p and there exists an m -nontrapping neighborhood $\langle p; q \rangle^*$ of p , then for all O_x (neighborhoods of x) and for all $z \in \langle p; q \rangle^*$ there is an $n \in \mathbb{N}$ such that $\langle p; z \rangle^* \subset f^{mn}(O_x)$.
- (ii) f has the shadowing property.

Proof. The proof is the straightforward consequence of the Main Theorem and Lemma 9 (if f is a map of type 2^n then f^{2^n} is a map with $\text{Per}(f^{2^n}) = \text{Fix}(f^{2^n})$). $\square$

Remark 2. The condition (i) is not fulfilled, in particular, when some critical point is eventually mapped in a “wrong way” (for “good way” see Figure 2) onto the repelling periodic point (see Figure 3).

However, this condition needn't be satisfied by a lot of maps from some family of maps. For instance, according to Lemma 7.3 in [1], for the family of tent maps the set of those parameters, for which the critical point is mapped onto a periodic point (which is in fact repelling), is dense in the set of all considered parameters.

In this section we prove that chaotic maps of the type 2^∞ do not have the shadowing property. We begin with

Definition 4. Let $f \in C^0(I, I)$. Then $u, v \in I$ are f -separable points if there are disjoint f -periodic intervals $J_u, J_v \subset I$ with $u \in J_u, v \in J_v$. Otherwise u, v are f -nonseparable.

Lemma 10. (Janková and Smítal [2]) Let $f \in C^0(I, I)$. Then f is chaotic if and only if there is an infinite attractor containing two f -nonseparable points.

Lemma 11. (Smítal [5]) Let $f \in C^0(I, I)$ be of type 2^∞ and let ω be an infinite attractor of f . Then there is a sequence $\{J_k\}_{k=0}^\infty$ of periodic intervals with the following properties:

- (i) J_k has period 2^k
- (ii) $J_{k+1} \subset J_k$
- (iii) $\omega \subset \bigcup_{i=1}^{2^k} f^i(J_k)$.

Theorem 3. Let $f \in C^0(I, I)$ be a chaotic map of the type 2^∞ . Then f does not have the shadowing property.

Proof. By Lemma 10 there is an infinite attractor ω containing two f -nonseparable points $p, q \in \omega$. By Lemma 11 there is a sequence $\{J_k\}_{k=0}^\infty$ which has the properties (i)–(iii). Without loss of generality let $p \in J_k$ for all k . If there is such $k \in \mathbb{N}$ that $q \notin J_k$ then p, q are f -separable which is a contradiction. Hence

$$(24) \quad p, q \in J_k \quad \text{for } k = 0, 1, \dots$$

Let $J = \bigcap_{k=0}^\infty J_k$. From Lemma 11 we obtain that

$$(25) \quad J \cap f^i(J) = \emptyset \quad \text{for } i \in \mathbb{N}.$$

Set $\varepsilon = |p - q|/2$, $\mathbf{x}_0 = (p + q)/2$, and let $\delta > 0$. Then by Lemma 11 there are $j, k \in \mathbb{N}$ such that

$$(26) \quad |f^j(J_k)| < \delta.$$

But there is $\mathbf{x}_j \in f^j(J_k)$ such that $f^{2^k-j}(\mathbf{x}_j) = \mathbf{x}_0$ (see (24)). Set

$$\begin{aligned} \mathbf{x}_i &= f^i(\mathbf{x}_0) \quad \text{for } 0 \leq i \leq j-1 \\ \mathbf{x}_{j+i} &= f^i(\mathbf{x}_j) \quad \text{for } 0 \leq i \leq 2^k - j. \end{aligned}$$

We have that $\{\mathbf{x}_i\}_{i=0}^{2^k}$ is a δ -chain (see (26)).

If $y \notin J$ then $|y - \mathbf{x}_0| > \varepsilon$ and if $y \in J$ then $|f^{2^k}(y) - \mathbf{x}_{2^k}| > \varepsilon$ (see (25)). Hence we find a δ -chain which is not ε -shadowed. $\square$

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