REGULAR MATRICES AND P-SETS IN $\beta N \setminus N$. II

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ABSTRACT. It was discovered by Henriksen and Isbell that the support in $\beta N \setminus N$ of a regular matrix is a P-set. We study conditions under which a P-subset of a matrix support set contains another matrix support set.

1. INTRODUCTION

Let $T = (t_{mn})$ be a nonnegative regular matrix. By regularity, T satisfies

- (i) $\sup_n \sum_k t_{nk} < \infty$,
- (ii) $\lim_{n \rightarrow \infty} \sum_k t_{nk} = 1$,
- (iii) $\lim_{k \rightarrow \infty} t_{nk} = 0$ for all n .

T defines a linear operator on the space $C_b(N)$ of all real-valued bounded functions on the positive integers N , by the formula $Tf(n) = \sum_b t_{nb} f(b)$. Classically, regular matrices were of interest because they served to extend the concept of *limit*: if $\lim_{n \rightarrow \infty} f(n) = c$ exists then $\lim_{n \rightarrow \infty} Tf(n) = c$, and this latter limit (called $T\text{-}\lim(f)$) may exist for many functions that do not have a limit in the usual sense.

If we restrict attention to functions of the form 1_A , where $A \subset N$, then we are able to make an interesting connection with the topology of $N^* = \beta N \setminus N$. Let F_T be the filter of subsets of N such that $T\text{-}\lim 1_A = 1$, and let K_T be the corresponding closed set in N^* , to be defined in the next section. K_T is called the *support* of T . In 1964 Henriksen and Isbell [HI] showed that K_T is a P-set, i.e., is interior to any closed G_δ set that contains it. (For the special case of the Cesaro matrix, this is even implicit in [H, p. 38].) In [A1] it is shown that, under the assumption N^* contains a dense set of P -points, every K_T set contains a family of 2^c pairwise disjoint P-sets, each the support of a regular Borel probability measure on N^* . (The support of a Borel probability measure is the intersection of all closed sets of measure 1.) Since a matrix support set cannot satisfy the countable chain condition [HI], it is clear that a P-set that is the support of a probability measure cannot be a matrix support set, so that under the continuum hypothesis ($C-H$), not every nowhere dense infinite P-set in N^* is the support of a matrix. (However it is consistent with ZFC that there are no c.c.c. P-sets in N^* [FSZ].) Just and Krawczyk [JK] showed

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that the supports of a large class of matrices are homeomorphic to each other, but not to N^* . More recently, Winfried Just has shown that no nowhere dense K_T set is homeomorphic to N^* , and he has also produced (without set theoretic hypotheses) a nowhere dense P-set in N^* that is not a K_T set. There still remains the problem of characterizing those nowhere dense P-sets that are supports of matrices.

In this paper we make a first effort at finding P-sets that contain matrix supports. If K_T is a support set and $L \subset K_T$ is a P-set, we give necessary and sufficient conditions for L to contain the support set of some submatrix of T .

There remains the problem of when L is *exactly* the support of a submatrix of T .

The general theory of P-sets in topological spaces has been developed by A. I. Veksler and his colleagues. (See, e.g., [V], and also [A2].) For a survey P-sets in N^* , see [M]. For compactification and summability see [FT].

2. PRELIMINARIES

We repeat some definitions from [A1]. If $A \subset N$, let A' be its closure in βN and $A^* = A' \cap (\beta N \setminus N) = A' \cap N^*$. Then $K_T = \bigcap \{A^* : A \in F_T\}$. If $f \in C_b(N)$, the space of bounded real functions on N , then f' is its extension to βN and $f^* = f'|N^*$.

$T = (t_{mn})$ is a positive regular matrix. If c_0 is the space of real functions on N vanishing at infinity then $T(c_0) \subset c_0$, so T induces an operator T^* on $C(N^*)$ by the formula $T^*f^* = (Tf)^*$. Also, $K_T = \bigcap \{A^* : T\text{-}\lim 1_A = 1\} = \bigcap \{A^* : A \in F_T\}$. We characterize K_T another way. If $p \in N^*$, let t_p be the Borel probability representing the functional $f \rightarrow T^*f(p)$ for $f \in C(N^*)$, so we have $T^*f(p) = \int f dt_p$. Then it is easy to see that $K_T = \text{closure} \cup \{K_p : p \in N^*\}$, where K_p is the support set of t_p . Below, L will be a P-set in N^* such that $L \subset K_T$. If $F' = \{A \subset N : L \subset A^*\}$ then $F_T \subset F'$.

If $f \in C_b(N)$, we write $Tf(n) = \sum_k t_{nk}f(k)$. If $f = 1_A$, the indicator of A , we may write $T1_A(n) = t_n(A)$.

A basic fact about N^* is that a nonvoid open set cannot be written as a union of cardinality c of closed nowhere dense sets [P, p. 46].

Main Theorem. *The following are equivalent:*

- (a) *there exists an infinite $A = \{n(k)\} \subset N$ such that for all $B \in F'$,*

$$\liminf t_{n(k)}(B) > 0;$$

- (b) *there exists an infinite $A = \{n(k)\}$ and infinite $E \subset N$ with*

$$\liminf t_{n(k)}(E) > 0,$$

such that L contains the support in N^ of the regular matrix operator*

$$Rf(k) = T(f1_E)(n(k))/T1_E(n(k)).$$

3. PROOF

“(b) implies (a).” Let F_R be the filter of sets summable to 1 by R , and let K_R be the corresponding closed set in N^* . If $L \supset K_R$ then $F' \subset F_R$, so for

$B \in F'$,

$$1 = \lim_{k \rightarrow \infty} \frac{t_{n(k)}(B \cap E)}{t_{n(k)}(E)} \leq \lim_{k \rightarrow \infty} \frac{t_{n(k)}(B)}{t_{n(k)}(E)}.$$

Since $\liminf t_{n(k)}(E) > 0$, it follows that $\liminf t_{n(k)}(B) > 0$, so (a) holds.

For “(a) implies (b),” we need some lemmas.

3.1. Lemma. *The function $t_p(L)$ defined for $p \in N^*$ is upper semicontinuous, so that $\{p : t_p(L) \geq t\}$ is closed for all real t .*

Proof. By regularity of the measure t_p ,

$$\begin{aligned} t_p(L) &= \inf\{t_p(A^*) : A \in F'\} \\ &= \inf\{T1_{A^*}(p) : A \in F'\}. \end{aligned}$$

Since each $T1_{A^*}$ is continuous on $C(N^*)$, the result follows.

3.2. Lemma. $\{p \in N^* : t_p(L) > 0\}$ has nonvoid interior.

Proof. Let A be as in hypothesis (a). We will prove A^* is contained in the set in question. Suppose not. Then there exists $p \in A^*$ such that $t_p(L) = 0$. By regularity of t_p , there exists for each n a $B_n \subset N$ such that $L \subset B_n^*$ and $t_p(B_n^*) < 1/n$. Since L is a P -set, there exists $B \in F'$ with $L \subset B^* \subset B_n^*$ for all n , so $t_p(B^*) = 0$. Since $p \in A^*$ and $A = (n(k))$, we get $\liminf t_{n(k)}(B) = 0$, contrary to (a).

3.3. Remark. If $L \subset K_T$, there always exists some p such that $t_p(L) > 0$. (*Proof.* Suppose $t_p(L) = 0$ for all $p \in N^*$. Since $t_p(L) = \lim\{t_p(A^*) : A \in F'\}$, a downward directed set of continuous functions, Dini's theorem implies the convergence is uniform. Hence for any n , there exists $A_n \in F'$ with $t_p(A_n^*) < 1/n$. Since F' is a P -filter, there exists $A \in F'$ with $A^* \subset A_n^*$ for all n , whence $t_p(A^*) = 0$ for all p . Since A^* is clopen, $A^* \cap K_T = \emptyset$, whence $L \cap K_T = \emptyset$, a contradiction.)

However there may not be “enough” p with $t_p(L) > 0$. For instance, if we let T be the identity matrix and L any P -set nowhere dense in N^* , then it is easy to see that no submatrix of T can have its support contained in L . Thus some hypothesis like (a) is really needed.

3.4. Lemma. For each $p \in N^*$, there exists $A \in F'$ such that $t_p(A^*) = t_p(L)$.

Proof. $t_p(L) = \inf\{t_p(A^*) : A \in F'\}$, so for each n , there exists $A_n \in F'$ with $t_p(A_n^*) < t_p(L) + 1/n$. If $A \in F'$ and $A^* \subset A_n^*$ for all n , then $t_p(L) \leq t_p(A^*) < t_p(L) + 1/n$ for all n .

3.5. Lemma. Let C be a nonvoid clopen subset of $\{p : t_p(L) > 0\}$. Then there exists $s > 0$ such that $C \cap \{p : t_p(L) \geq s\}$ has nonvoid interior.

Proof. Let $R_k = \{p : t_p(L) \geq 1/k\}$, which is closed by 3.1. Then $C = \bigcup_k (R_k \cap C)$, so by Baire category, there exists k so that $R_k \cap C$ has nonvoid interior.

3.6. Lemma. There exists nonvoid clopen $A^* \subset N^*$ and $E \in F'$ such that for all $p \in A^*$, $t_p(E^*) = t_p(L) \geq s$, where s is as in 3.5.

Proof. Let C and s be as in 3.5, and let W be a clopen subset of $C \cap \{p : t_p(L) \geq s\}$. By 3.4, for each $p \in W$, there exists $A_p \in F'$ with $t_p(L) = t_p(A_p^*) \geq s$. Let

$$Z_p = \{q \in W : t_q(A_p^*) = t_q(L)\},$$

a closed set. (Note that $t_q(L) - t_q(A_p^*) \leq 0$ and is upper semicontinuous. Now $\{q : t_q(L) - t_q(A_p^*) = 0\} = \{q : t_q(L) - t_q(A_p^*) \geq 0\}$, so this set is closed.)

Since $W = \bigcup \{Z_p : p \in W\}$ and a clopen set in N^* is not the union of c nowhere dense closed sets, at least one of the Z_p has nonvoid interior. (The number of distinct Z_p sets is at most c because this is true of the corresponding A_p sets.) Let $A \subset N$ be such that $A^* \subset Z_p$, and let E be the element A_p corresponding to Z_p .

3.7. Proof that (a) implies (b). Let $A = \{n(k)\}$ and E be as in 3.6. If the matrix operator R is defined as in the statement of the main theorem, then it is clear that for all $B \in F'$, $R\text{-}\lim 1_B = 1$, and it follows easily that the support of R is contained in L . This completes the proof.

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