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ON OPIAL'S INEQUALITY FOR $f^{(n)}$

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ABSTRACT. We prove inequalities of the type

$$\int_0^h |f^{(i)}(x)f^{(j)}(x)|\,dx \le C(n, i, j, p)h^{2n-i-j+1-2/p}\left(\int_0^h |f^{(n)}(x)|^p\,dx\right)^{2/p}$$

when $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. We assume that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_p(0, h)$, with $p \ge 1$, $n \ge 2$, and $0 \le i \le j \le n-1$.

Opial's inequality is $\int_0^h |f(x)f'(x)|\,dx \le \frac{h}{4}\int_0^h f'(x)^2\,dx$ when $f(0) = f(h) = 0$. Shortly after this result was published it was realized that the inequality followed from two applications of the inequality

$$\int_0^h |f(x)f'(x)|\,dx \le \frac{h}{2}\int_0^h f'(x)^2\,dx$$

when $f(0) = 0$. Various generalizations of this result have been given, see Mitrinović [2] or Mitrinović, Pečarić, and Fink [3]. Here we look at the inequality

(1)
$$\int_0^h |f^{(i)}(x)f^{(j)}(x)|\,dx \le C(n, i, j, p)h^{2n-i-j+1-2/p}\left(\int_0^h |f^{(n)}(x)|^p\,dx\right)^{2/p}$$

when $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. We assume that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_p(0, h)$, and $p \ge 1$, $n \ge 2$, and $0 \le i \le j \le n-1$.

We begin with the statement of the

Theorem. *Let f have $(n-1)$ continuous derivatives and assume that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_p(0, h)$. Let f have an n-fold zero at 0. Then inequality (1) holds with best possible constants $C(n, i, j, p)$ given by*

$$C(n, i, i+1, p) = \frac{1}{2(n-i-1)!^2[(n-i-1)p'+1]^{2/p'}} \tag{2}$$

for $i = 0, \ldots, n-2$, while

$$C(n, i, j, p) \le \frac{2^{-1/p}}{(n-i-1)!(n-j)![(n-j)p'+1]^{1/p'}[(2n-i-j-1)p'+2]^{1/p'}} \tag{3}$$

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for $0 \le i \le j \le n-1$. *Equality holds in* (1) *if*

$$f(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}(h-t)^{(n-i-1)p'/p}\,dt, \qquad p > 1.$$

The proof of this theorem is based on the representation

$$(4)\quad f(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f^{(n)}(t)\,dt = \frac{1}{(n-1)!}\int_0^h (x-t)_+^{n-1} f^{(n)}(t)\,dt,$$

where $x_+ = \max(0, x)$. We will give two proofs of (2), the first because it gives a guide to the proof of (3). The second is given at the end of the paper and is simpler.

Using (4) we can easily derive

$$f^{(i)}(x)f^{(j)}(x) = \frac{1}{(n-1-i)!(n-1-j)!} \times \int_0^h\int_0^h f^{(n)}(t)f^{(n)}(s)[(x-t)_+^{n-1-i}(x-s)_+^{n-1-j}]\,dt\,ds.$$

We prefer to symmetrize this to get

(5)

$$f^{(i)}(x)f^{(j)}(x) = \frac{1}{2(n-1-i)!(n-1-j)!}\int_0^h\int_0^h f^{(n)}(t)f^{(n)}(s)k(x, t, s)\,ds\,dt,$$

where

$$(6)\qquad k(x, t, s) \equiv (x-t)_+^{n-1-i}(x-s)_+^{n-1-j} + (x-s)_+^{n-1-i}(x-t)_+^{n-1-j}.$$

Proof of (2). We use relation (5) to get

$$(7)\qquad \begin{aligned} &\int_0^h |f^{(i)}(x)f^{(i+1)}(x)|\,dx \\ &\quad \le \frac{1}{2(n-1-i)!(n-i-2)!}\int_0^h\int_0^h |f^{(n)}(t)|\,|f^{(n)}(s)|\left(\int_0^h k(x, t, s)\,dx\right)dt\,ds. \end{aligned}$$

We note that equality holds if $f^{(n)}$ is of constant sign since this implies that $f^{(i)}f^{(i+1)} \ge 0$ and $k \ge 0$. Now the kernel $k(x, t, s)$ depends on i and j also and we have in the case $j = i+1$ and $t \ge s$

$$\begin{aligned} \int_0^h k(x, t, s)\,dx &= \int_t^h (x-t)^{n-i-2}(x-s)^{n-i-2}[(x-t)+x-s]\,dx \\ &= \int_t^h \frac{d}{dx}\left(\frac{[(x-t)(x-s)]^{n-i-1}}{n-i-1}\right)dx = \frac{[(h-t)(h-s)]^{n-i-1}}{n-i-1}. \end{aligned}$$

By symmetry this formula also holds for $t \le s$.

The integral on the right-hand side of (7) becomes

$$(8)\qquad \frac{\left(\int_0^h (h-t)^{n-i-1}|f^{(n)}(t)|\,dt\right)^2}{(n-i-1)}.$$

We estimate this by Hölder's inequality to get an upper bound to (8)

$$(9)\qquad \frac{\left(\int_0^h (h-t)^{(n-i-1)p'}\,dt\right)^{2/p'}\left(\int_0^h |f^{(n)}(t)|^p\,dt\right)^{2/p}}{(n-i-1)} = \frac{h^{((n-i-1)p'+1)2/p'}}{(n-i-1)((n-i-1)p'+1)^{2/p'}}\left(\int_0^h |f^{(n)}(t)|^p\,dt\right)^{2/p}.$$

We have equality if $|f^{(n)}_{(t)}|^p = (h-t)^{n-i-1)p'}$ a.e. In particular we have the integral on the right-hand side of (7) equal to (9) when $f^{(n)}(t) = (h-t)^{(n-i-1)p'/p}$. At the same time this gives equality in (7) as noted above. Furthermore, equality holds only if f is a multiple of one such extremal. The cases $p = 1$ and ∞ need to be taken care of separately. For $p = \infty$ we easily get (2) by the obvious estimate of (8). For $p = 1$ we take the factor in (2) that contains p' to be 1. Equality holds when $p = \infty$ when $f^{(n)}(t) = \text{constant}$. For $p = 1$ we of course never get equality. This completes the proof of (2).

The case when $j \neq 1+i$ differs from the above case in that $\int_0^h k(x, t, s)\,dx$ does not factor into a product of a power of $(1-t)$ and a power of $(1-s)$. We must be content to bound it with such a function. In preparation for this argument we offer a

Lemma. *For $u \geq 0$ and $0 \leq i \leq j$*

$$(10)\qquad \int_0^u [w^i(1+w)^j + w^i(1+w)^i]\,dw \leq \frac{u^{i+1}(1+u)^j}{i+1},$$

and the number $1/(i+1)$ cannot be replaced by a smaller constant.

Proof. We write the right-hand side of (10) as

$$\frac{1}{1+i}\int_0^u \frac{d}{dw}(w^{i+1}(1+w)^j)\,dw$$

and find that this integrand dominates the integrand on the left of (10) as long as $j \geq i$ and $w \geq 0$. To show that the constant $1/(1+i)$ cannot be replaced by a smaller one, one argues that

$$\lim_{u\to 0^+} \frac{1}{u^{i+1}(1+u)^j}\int_0^u [w^i(1+w)^j + w^j(1+w)^i]\,dw = \frac{1}{1+i}$$

prevents this.

We may now prove the remainder of the theorem.

Proof of (3). We begin with the inequality like (7) obtained from (5) and consider a bound on $\int_0^h k(x, t, s)\,dx$. In this integral we assume first that $x > t > s$ and make the change of variables $w = (x-t)/(t-s)$. By use of the lemma, we obtain

$$\begin{aligned}\int_0^h k(x, t, s)\,dx &= (t-s)^{2n-i-j-1}\int_0^{(h-t)/(t-s)} [w^{n-i-1}(1+w)^{n-j-1} + w^{n-j-1}(1+w)^{n-i-1}]\,dw \\ &\leq \frac{(t-s)^{2n-i-j-1}}{n-j}\left(\frac{h-t}{t-s}\right)^{n-j}\left(\frac{h-s}{t-s}\right)^{n-i-1} = \frac{(h-t)^{n-j}(h-s)^{n-i-1}}{n-j}.\end{aligned}$$

Recall that this is for $t > s$. By continuity the final estimate holds when $t = s$. By symmetry, we may estimate the integral for $s > t$.

The integral on the right of (7) can now be estimated by

$$\begin{aligned}
&\frac{1}{n-j}\iint_{t\geq s}(h-t)^{n-j}(h-s)^{n-1-i}|f^{(n)}(t)||f^{(n)}(s)|\,dt\,ds \\
&\quad+\frac{1}{n-j}\iint_{t\leq s}(h-s)^{n-j}(h-t)^{n-i-1}|f^{(n)}(t)||f^{(n)}(s)|\,dt\,ds \\
&=\frac{2}{n-j}\int_0^h(h-s)^{n-i-1}|f^{(n)}(s)|\left(\int_s^h(h-t)^{n-j}|f^{(n)}(t)|\,dt\right)ds \\
&\leq\frac{2}{n-j}\int_0^h(h-s)^{n-1-i}|f^{(n)}(s)|\left(\int_s^h(h-t)^{(n-j)p'}\,dt\right)^{1/p'}\left(\int_s^h|f^{(n)}(t)|^p\,dt\right)^{1/p} \\
&=\frac{2}{(n-j)[(n-j)p'+1]^{1/p'}}\int_0^h(h-s)^{2n-i-j-1/p}|f^{(n)}(s)| \\
&\qquad\qquad\times\left(\int_s^h|f^{(n)}(t)|^p\,dt\right)^{1/p}ds \\
&\leq\frac{2}{(n-j)[(n-j)p'+1]^{1/p'}}\left(\int_0^h(h-s)^{(2n-i-j-1/p)p'}\,ds\right)^{1/p'} \\
&\quad\times\left(\int_0^h|f^{(n)}(s)|^p\int_s^h|f^{(n)}(t)|^p\,dt\,ds\right)^{1/p} \\
&=\frac{2h^{2n-i-j+1-2/p}}{(n-j)[(n-j)p'+1]^{1/p'}(2n-i-j-1)p'+2]^{1/p'}} \\
&\quad\times\left[\frac{1}{2}\left(\int_0^h|f^{(n)}(t)|^p\,dt\right)^2\right]^{1/p}.
\end{aligned}$$

This gives the final result for $1 < p < \infty$. One can easily check that for $p = 1$ or ∞ the result is also valid.

A related result is given by Fitzgerald [1], who considers the inequality (1) for the case $i = 0$, $j = 1$, $p = 2$ with the boundary conditions that f has an n-fold zero at both 0 and h. He gets the extremal to be a piecewise polynomial and is able to compute the constant explicit for small n. For $n = 2$ he gets $\frac{1}{192}$ while we have the best possible constant $\frac{1}{6}$. The case $j = i+1$ has an easy proof as mentioned above. For if f is given, let

$$g(x) = \frac{1}{(n-1)!}\int_0^x(x-t)^{n-1}|f^{(n)}(t)|\,dt.$$

Then $g^{(i)}(x) \geq 0$ for all i and $|f^{(i)}(t)| \leq g^i(x)$ for all i and x, with $|f^{(n)}(x)| = g^{(n)}(x)$.

Then

$$\int_0^h |f^{(i)}(x)f^{(i+1)}(x)|\,dx \le \int_0^h g^{(i)}(x)g^{(i+1)}(x)\,dx$$
$$= \frac{(g^{(i)}(h))^2}{2} = \frac{1}{2}\left(\int_0^h \frac{(h-t)^{n-i}}{(n-i)!}|f^{(n)}(t)|\,dt\right)^2,$$

and one applies Hölder's inequality as in the first proof.

References

1. C. H. Fitzgerald, *Opial-type inequalities that involve higher order derivatives*, General Inequalities, 4th ed., W. Walter, Basel, 1984, pp. 25–36.
2. D. S. Mitrinović, *Analytic inequalities*, Springer-Verlag, New York, 1970.
3. D. S. Mitrinović, J. T. Pečarić, and A. M. Fink, *Inequalities for functions involving their integrals and derivatives*, Kluwer, Dordrecht, 1991.
4. Z. Opial, *Sur une inégalité*, Ann. Polon. Math. **8** (1960), 29–32.

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