CHARACTERIZATION OF SEPARABLE METRIC $\mathbf{R}$ -TREES

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ABSTRACT. An $\mathbf{R}$ -tree (X, d) is a uniquely arcwise connected metric space in which each arc is isometric to a subarc of the reals. $\mathbf{R}$ -trees arise naturally in the study of groups of isometries of hyperbolic space. Two of the authors had previously characterized $\mathbf{R}$ -trees topologically among metric spaces. The purpose of this paper is to provide a simpler proof of this characterization for separable metric spaces. The main theorem is the following: *Let (X, r) be a separable metric space. Then the following are equivalent:*

- (1) X admits an equivalent metric d such that (X, d) is an $\mathbf{R}$ -tree.
- (2) X is locally arcwise connected and uniquely arcwise connected.

The method of proving that (2) implies (1) is to "improve" the metric r through a sequence of equivalent metrics of which the first is monotone on arcs, the second is strictly monotone on arcs, and the last is convex, as desired.

1. INTRODUCTION

1.1. $\mathbf{R}$ -trees. An $\mathbf{R}$ -tree (X, d) is a uniquely arcwise connected metric space (see definition below) in which each arc is isometric to a subarc of the reals. ("Uniquely" is superfluous in this definition.) Actions on $\mathbf{R}$ -trees can be seen as ideal points in the compactification of groups of isometries of hyperbolic space [Mr, Be, MrS]. The $\mathbf{R}$ -trees considered in these applications are always separable. In this paper we provide a much simpler proof of the characterization of $\mathbf{R}$ -trees among separable metric spaces than the proof provided for metric spaces in general in [MO].

1.2. Main Theorem (Characterization Theorem). (See Theorem 5.1 of [MO].) *Let (X, r) be a separable metric space. Then the following are equivalent:*

- (1) X admits an equivalent metric d such that (X, d) is an $\mathbf{R}$ -tree.
- (2) X is locally arcwise connected and uniquely arcwise connected.

If r is a metric on a space X , by $B_r(x, \varepsilon)$ we denote the open ball about $x \in X$ of radius $\varepsilon > 0$ in the metric r . That (1) implies (2) follows from the

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observation that in an $\mathbf{R}$ -tree (X, d) , if $d(x, z) = \delta$, then there is an arc from x to z , at each point $y \neq z$ of which $d(x, y) < \delta$. (See the definition of a convex metric and Proposition 1.4 below.) Hence, for all $\varepsilon > 0$, $B_d(x, \varepsilon)$ is arcwise connected.

The proof that (2) implies (1) proceeds through a series of lemmas, which we present in §2. These lemmas gradually “improve” the metric r on X through a sequence of equivalent metrics ρ , ρ^* , and d , such that ρ is monotone on arcs, ρ^* is strictly monotone on arcs, and, finally, d is convex, so that (X, d) is an $\mathbf{R}$ -tree.

1.3. Definitions. An *arc* A in a space X is the image $A = e([0, 1])$ of an embedding $e: [0, 1] \rightarrow X$; the *end points* of A are $e(0)$ and $e(1)$. We say a space X is (uniquely) *arcwise connected* iff given $x \neq y \in X$, there is a (unique) arc $A \subset X$ whose end points are x and y . In a uniquely arcwise connected space, the intersection of any two arcwise connected subsets is again arcwise connected.

Let X be a uniquely arcwise connected space with $x \neq y \in X$. By $[x, y]$ we denote the unique arc in X whose end points are x and y ; $[x, x]$ denotes $\{x\}$. Let $p \in X$ be fixed. Since X is uniquely arcwise connected, we have for all $x, y \in X$, $[p, x] \cap [p, y] = [p, z]$ for some $z \in X$ (possibly, $[p, z]$ is degenerate). Define a *meet* function $\wedge_p: X \times X \rightarrow X$ with respect to p by setting $x \wedge_p y = z$, where z is defined as above. We will drop the p subscript where this can be done without confusion.

We say a space X is *locally arcwise connected* iff for every point $x \in X$ and every open neighborhood U of x , there is an arcwise connected open neighborhood V of x with $V \subset U$.

Let (X, r) be a uniquely arcwise connected metric space. We say that r is *monotone on arcs* (respectively, *strictly monotone on arcs*) iff for all $x \neq z \in X$, for every $y \in [x, z]$, $r(x, y) \leq r(x, z)$ (respectively, $r(x, y) < r(x, z)$). We call the metric r on X a *convex metric* iff for all $x, z \in X$, for all $y \in [x, z]$, $r(x, z) = r(x, y) + r(y, z)$. A convex metric on X is strictly monotone on arcs, though not necessarily conversely. The following equivalence is easy to prove:

1.4. Proposition. *Let (X, d) be a uniquely arcwise connected metric space. Then the following are equivalent:*

- (1) (X, d) is an $\mathbf{R}$ -tree.
- (2) d is a convex metric on X .

2. IMPROVING THE METRIC

In this section we assume that (X, r) is a uniquely arcwise connected, locally arcwise connected, separable metric space. We may assume without loss of generality that r is bounded. In the following lemmas we show how to improve the metric r on X , through a series of equivalent metrics, to a metric d on X with respect to which X is an $\mathbf{R}$ -tree.

2.1. Lemma. *The metric ρ on X , defined by $\rho(x, y) = \text{diam}_r([x, y])$, is bounded, equivalent to r , and monotone on arcs.*

Proof. If $y \in [x, z]$, then $[x, y] \subset [x, z]$, which implies $\text{diam}_r([x, y]) \leq \text{diam}_r([x, z])$; so ρ is monotone on arcs and bounded by the bound of r .

That ρ is a metric is easy to verify. We verify the triangle inequality: since X is uniquely arcwise connected and $[x, y] \cap [y, z] \neq \emptyset$, we have $[x, z] \subset [x, y] \cup [y, z]$. Thus,

$$\text{diam}_r([x, z]) \leq \text{diam}_r([x, y] \cup [y, z]) \leq \text{diam}_r([x, y]) + \text{diam}_r([y, z]).$$

Hence, $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

Since $\rho(x, y) = \text{diam}_r([x, y]) \geq r(x, y)$, we have $B_\rho(x, \varepsilon) \subset B_r(x, \varepsilon)$; so the topology on X generated by ρ is finer than the topology on X generated by r . Conversely, to see that r is finer than ρ , let $\varepsilon > 0$ and $x \in X$ be given. Since X is locally arcwise connected, there is an arcwise connected r -open neighborhood U of x such that $U \subset B_r(x, \varepsilon/2)$. Then for all $y \in U$, $\rho(x, y) = \text{diam}_r([x, y]) < \varepsilon$. Therefore, $U \subset B_\rho(x, \varepsilon)$. $\square$

2.2. Lemma. *Let D be a countable dense subset of X , and let $\{J_i\}_{i=1}^\infty$ be the collection of all arcs between different points of D . For each i , let d_i be a convex metric on J_i such that $\text{diam}_{d_i}(J_i) = 2^{-i}$. Then the metric ρ^* on X , defined by*

$$\rho^*(x, y) = \rho(x, y) + \sum_{i=1}^{\infty} \text{diam}_{d_i}(J_i \cap [x, y]),$$

is bounded, equivalent to ρ , and strictly monotone on arcs.

Proof. From the fact that X is uniquely arcwise connected, it is easy to show that the following hold for all $x, y, z \in X$, for all i :

- (1) $J_i \cap [x, y]$ is either empty, a point, or an arc,
- (2) $[x, z] \subset [x, y] \cup [y, z]$, and
- (3) $J_i \cap ([x, y] \cup [y, z])$ is either empty, a point, or an arc.

Consequently, we have

$$J_i \cap [x, z] \subset J_i \cap ([x, y] \cup [y, z]) = (J_i \cap [x, y]) \cup (J_i \cap [y, z])$$

and thus,

$$\text{diam}_{d_i}(J_i \cap [x, z]) \leq \text{diam}_{d_i}(J_i \cap [x, y]) + \text{diam}_{d_i}(J_i \cap [y, z]).$$

The triangle inequality now follows from the definition of ρ^* by considering each term in the sum. The remaining properties of a metric are easy to check. The bound on ρ^* is the bound on ρ plus 1.

Since the set of end points of the J_i 's is dense in X and X is locally arcwise connected and uniquely arcwise connected, it follows that if U and V are disjoint closed neighborhoods of x and y , respectively, then there is some index i such that for $J_i = [a, b]$, we have $a, x \wedge_y a \in U$ and $b, y \wedge_x b \in V$; so $J_i \cap [x, y] \supset [x, y] - (U \cup V)$. In particular, $J_i \cap [x, y]$ is nondegenerate. (It may be helpful to draw a figure.) It now follows from the definition of ρ^* that for all $x \neq y \in X$, $\rho^*(x, y) > \rho(x, y)$. Therefore, ρ^* is finer than ρ .

Now suppose $y \in [x, z]$. It follows that for all i ,

$$(*) \quad \text{diam}_{d_i}(J_i \cap [x, y]) \leq \text{diam}_{d_i}(J_i \cap [x, z]).$$

But by the argument in the preceding paragraph (choose U about x and V about z to both miss y), there is an index i such that strict inequality obtains in (*), since d_i is convex, hence strictly monotone, on J_i . It follows that $\rho^*(x, y) < \rho^*(x, z)$; so, ρ^* is strictly monotone on arcs.

To see that ρ is finer than ρ^* , let $\varepsilon > 0$ and $x \in X$ be given. Choose N such that $\sum_{i=N+1}^{\infty} 2^{-i} < \varepsilon/2$. For $i = 1, 2, \dots, N$, partition each J_i into subarcs of d_i -diameter at most $\varepsilon/8N$. Since X is locally arcwise connected, we may choose an arcwise connected ρ -open neighborhood U of x such that $U \subset B_\rho(x, \varepsilon/4)$ and each arc in U meets J_i , for $i = 1, 2, \dots, N$, in at most two partition elements. Thus, if $y \in U$, then $\text{diam}_{d_i}(J_i \cap [x, y]) \leq \varepsilon/4N$. We then have

$$\begin{aligned} \rho^*(x, y) &= \rho(x, y) + \sum_{i=1}^N \text{diam}_{d_i}(J_i \cap [x, y]) + \sum_{i=N+1}^{\infty} \text{diam}_{d_i}(J_i \cap [x, y]) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore, $U \subset B_{\rho^*}(x, \varepsilon)$. $\square$

In the following lemma, note that we are not claiming that the metric d_p is equivalent to ρ^* on X .

2.3. Lemma. *Let ρ^* be a bounded metric on X strictly monotone on arcs, and let $p \in X$. For all $x, y \in X$, define*

$$d_p(x, y) = \rho^*(p, x) + \rho^*(p, y) - 2\rho^*(p, x \wedge y).$$

Then d_p is a convex metric on X bounded by twice the bound on ρ^ .*

Proof. Throughout the proof, $\wedge$ will denote $\wedge_p$. Since $x \wedge y = y \wedge x$, d_p is symmetric. Since $x \wedge y \in [p, x]$ and $x \wedge y \in [p, y]$ and ρ^* is monotone on arcs, we have $\rho^*(p, x) - \rho^*(p, x \wedge y) \geq 0$ and $\rho^*(p, y) - \rho^*(p, x \wedge y) \geq 0$. Consequently, $d_p(x, y) \geq 0$. If $x = y$ then $x = x \wedge y = y$; so $d_p(x, y) = 0$. Conversely, if $d_p(x, y) = 0$, then

$$\rho^*(p, x) - \rho^*(p, x \wedge y) = 0 \quad \text{and} \quad \rho^*(p, y) - \rho^*(p, x \wedge y) = 0$$

since their sum is 0 and both are nonnegative. Thus,

$$\rho^*(p, x) = \rho^*(p, x \wedge y) \quad \text{and} \quad \rho^*(p, y) = \rho^*(p, x \wedge y).$$

Since ρ^* is strictly monotone on arcs, this implies $x = x \wedge y = y$. That d_p is bounded by twice the bound on ρ^* is clear.

To establish the triangle inequality, suppose $x, y, z \in X$. Let $v = x \wedge z$ and let b be the last point of the arc $[p, y]$ that lies in $[p, x] \cup [p, z]$. There are three cases: $b \in [p, v]$, $b \in (v, z]$, and $b \in (v, x]$. (The reader may find it helpful to draw some figures.) We do the first case below and leave the similar proofs of the remaining cases to the reader.

If $b \in [p, v]$, then $x \wedge y = b = y \wedge z$, $\rho^*(p, y) \geq \rho^*(p, b)$, and $\rho^*(p, v) \geq \rho^*(p, b)$. Thus we have

$$\begin{aligned} d_p(x, y) + d_p(y, z) &= \rho^*(p, x) + 2\rho^*(p, y) + \rho^*(p, z) - 4\rho^*(p, b) \\ &\geq \rho^*(p, x) + \rho^*(p, z) - 2\rho^*(p, b) \\ &\geq \rho^*(p, x) + \rho^*(p, z) - 2\rho^*(p, v) = d_p(x, z). \end{aligned}$$

To verify that d_p is convex, let A be an arc in X and let $x, y, z \in A$ such that $y \in [x, z]$. Without loss of generality, assume that $x \wedge z = v \in [y, z]$. Then $x \wedge y = y$ and $y \wedge z = v$. Thus we have

$$\begin{aligned} d_p(x, y) &= \rho^*(p, x) + \rho^*(p, y) - 2\rho^*(p, y) = \rho^*(p, x) - \rho^*(p, y), \\ d_p(y, z) &= \rho^*(p, y) + \rho^*(p, z) - 2\rho^*(p, v), \end{aligned}$$

$$d_p(x, y) + d_p(y, z) = \rho^*(p, x) + \rho^*(p, z) - 2\rho^*(p, v) = d_p(x, z). \quad \square$$

2.4. Example. The following example shows that it is not generally the case that if ρ^* is a metric strictly monotone on arcs and p is an arbitrary point in X , then d_p , defined as in Lemma 2.3, is equivalent to ρ^* . However, it does follow by the argument in paragraph three of §2.5 that ρ^* is finer than d_p .

Let p be the origin $(0, 0)$, x the point $(0, 2)$, and $[p, x]$ the line segment joining them in the plane. Let $a_i = (0, 2 - \frac{1}{i})$, $x_i = (1, 2 - \frac{1}{i})$, and let $[a_i, x_i]$ be the line segment joining them. Our space is

$$X = [p, x] \cup \left(\bigcup_{i=1}^{\infty} [a_i, x_i] \right).$$

The idea is to define a metric ρ^* on X , strictly monotone on arcs, that has the property that from the point of view of p the arcs $[a_i, x_i]$ look short, but from the point of view of x they look long.

Let d denote the standard (Euclidean) metric on the plane, with π_1 and π_2 denoting the projections to the horizontal and vertical axes, respectively. Let d_1 denote the "arc-length" metric on X ; that is, the d_1 -distance between two points $a, b \in X$ is the Euclidean arc-length of the unique arc $[a, b]$ between them. Let d_2 be defined as follows:

$$d_2(b, a) = d_2(a, b) = \begin{cases} d(a, b), & \text{if } a, b \in [p, x], \\ d(a, a_i) + \frac{d(a_i, b)}{i} & \text{if } a \in [p, x], \pi_2(b) = \pi_2(a_i). \end{cases}$$

Note that d_2 is only defined on a subset of $X \times X$.

The metric ρ^* on X is defined as follows:

$$\rho^*(b, a) = \rho^*(a, b) = \begin{cases} d_1(a, b), & \text{if } a, b \in [p, a_1] \text{ or } a, b \in X - [p, a_1], \\ td_1(a, b) + (1-t)d_2(a, b), & \text{if } a \in [p, a_1], b \in X - [p, a_1], t = \pi_2(a). \end{cases}$$

We show below that ρ^* is a metric on X , strictly monotone on arcs, and that the convex metric d_p on X defined with respect to ρ^* and p as in Lemma 2.3 has the property that $d_p(x, x_i) \rightarrow 0$. Since $\rho^*(x, x_i) = d_1(x, x_i) > 1$, for all i , this shows d_p is not equivalent to ρ^* .

It is easy to see that as a function from $X \times X$ to $\mathbf{R}$, ρ^* is nonnegative, symmetric, and takes value 0 if and only if $a = b$. Moreover, on $[p, x]$ and on $X - [p, a_1]$, ρ^* reduces to the arc-length metric d_1 .

To see that ρ^* is strictly monotone on arcs, suppose $a, b \in X$. If a, b are both in $[p, x]$ or $X - [p, a_1]$, then $\rho^* = d_1$; hence, ρ^* is strictly monotone on $[a, b]$. So we may suppose $a \in [p, a_1]$ and $b \in (a_i, x_i]$, for some i . Then (by abuse of notation) we see that $2 > \pi_2(b) = \pi_2(a_i) = a_i \geq 1$, $0 \leq \pi_2(a) = a < 1$, and $0 < \pi_1(b) = b \leq 1$. One may then compute that

$$\begin{aligned} \rho^*(a, b) &= ad_1(a, b) + (1-a)d_2(a, b) \\ &= a[d(a, a_i) + d(a_i, b)] + (1-a) \left[d(a, a_i) + \frac{d(a_i, b)}{i} \right] \\ &= a_i - a + b \left(a + \frac{1-a}{i} \right) \quad \text{written as a function of } b \\ &= a_i + \frac{b}{i} + a \left(-1 + b - \frac{b}{i} \right) \quad \text{written as a function of } a. \end{aligned}$$

It is then easy to see that for fixed a , as b increases (i.e., $[a, b]$ gets longer in the b -direction), $\rho^*(a, b)$ increases, while for fixed b , as a decreases (i.e., $[a, b]$ gets longer in the a -direction), $\rho^*(a, b)$ increases. Therefore, ρ^* is strictly monotone on arcs.

Note that for $a = 1$, $b = 0$, or $i = 1$, $\rho^*(a, b)$ reduces to the arc-length of $[a, b]$: $d_1(a, b) = a_i - a + b$. The above arguments show that for all $a, b \in X$, $\rho^*(a, b) \leq d_1(a, b)$, with equality holding if either $a, b \in [p, x] \cup [a_1, x_1]$ or $a, b \in X - [p, a_1]$.

It remains to show that ρ^* satisfies the triangle inequality. So suppose $a, b \in X$. Since ρ^* is strictly monotone on arcs, it suffices to check the triangle inequality for $c \in [a, b]$. If $[a, b] \subset [p, x]$ or $[a, b] \subset X - [p, a_1]$, then $\rho^* = d_1$; so we may suppose $a \in [p, a_1)$ and $b \in (a_i, x_i]$, for some i . Let $c \in [a, b]$. As above, assume $\pi_2(a_i) = a_i$, $\pi_1(b) = b$, and $\pi_2(a) = a$. There are three cases to consider.

If $c \in [a_1, a_i]$, then we have

$$\rho^*(a, b) \leq d_1(a, b) \leq d_1(a, c) + d_1(c, b) = \rho^*(a, c) + \rho^*(c, b).$$

Suppose $c \in [a, a_1]$; then $0 \leq a \leq \pi_2(c) = c \leq 1$. Observe that

$$\begin{aligned} \rho^*(a, b) &= a_i - a + \frac{b}{i} + a \left(b - \frac{b}{i} \right), \\ \rho^*(a, c) + \rho^*(c, b) &= a_i - a + \frac{b}{i} + c \left(b - \frac{b}{i} \right). \end{aligned}$$

Since $c \geq a$, the triangle inequality holds.

Finally, suppose $c \in [a_i, b]$; then $0 \leq \pi_1(c) = c \leq b \leq 1$. Observe that

$$\begin{aligned} \rho^*(a, b) &= a_i - a + b \left(a + \frac{1-a}{i} \right), \\ \rho^*(a, c) + \rho^*(c, b) &= a_i - a + b + c \left(-1 + a + \frac{1-a}{i} \right). \end{aligned}$$

The equation for $\rho^*(a, c) + \rho^*(c, b)$ is linear in c ; when $c = 0$, it reduces to

$$\rho^*(a, c) + \rho^*(c, b) = a_i - a + b = d_1(a, b) \geq \rho^*(a, b);$$

when $c = b$, it reduces to

$$\rho^*(a, c) + \rho^*(c, b) = \rho^*(a, b).$$

So for $0 \leq c \leq b$, the triangle inequality holds.

Now ρ^* is a metric on X , strictly monotone on arcs. We define the convex metric d_p with respect to ρ^* and point $p \in X$ as in Lemma 2.3. Then one may compute

$$d_p(x, x_i) = 2 - d(p, a_i) + \frac{d(a_i, x_i)}{i} = \frac{2}{i} \rightarrow 0.$$

2.5. Proof of Theorem 1.2. Let (X, r) be a uniquely arcwise connected, locally arcwise connected, separable metric space. Without loss of generality, suppose that r is bounded. Let ρ^* be the bounded metric of Lemma 2.2 equivalent to r and strictly monotone on arcs. In order to obtain a convex metric on X equivalent to r , we must again use the separability of X . Let $\{p_i\}_{i=1}^\infty$ be a

countable dense set in X , and for each i , let $d_i = d_{p_i}$ be the convex metric of Lemma 2.3 defined in terms of ρ^* and p_i . Note that there is a uniform bound on the d_i 's.

For all $x, y \in X$, define

$$d(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x, y)}{2^i}.$$

It is easy to see that d is a bounded convex metric on X . So it remains to show that d is equivalent to ρ^* .

To see that ρ^* is finer than d , observe that if $\rho^*(x_j, x) \xrightarrow{j} 0$, then, since X is locally arcwise connected, $\rho^*(x_j \wedge_{p_i} x, x) \xrightarrow{j} 0$ for each i . Consequently, by the triangle inequality and the continuity of ρ^* , we have for each i ,

$$\begin{aligned} \rho^*(p_i, x) - \rho^*(p_i, x_j \wedge_{p_i} x) &\xrightarrow{j} 0, \\ \rho^*(p_i, x_j) - \rho^*(p_i, x_j \wedge_{p_i} x) &\xrightarrow{j} 0. \end{aligned}$$

Since

$$d_i(x_j, x) = \rho^*(p_i, x_j) - \rho^*(p_i, x_j \wedge_{p_i} x) + \rho^*(p_i, x) - \rho^*(p_i, x_j \wedge_{p_i} x),$$

it follows that $d_i(x_j, x) \xrightarrow{j} 0$ for each i . Since the sum converges, $d(x_j, x) \xrightarrow{j} 0$.

To prove that d is finer than ρ^* , we show that for each $x \in X$ and each $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho^*(x, y) < \varepsilon$. Let $\varepsilon > 0$ and $x \in X$ be given. Choose n so that $\rho^*(p_n, x) < \varepsilon/3$. Choose $\delta = \varepsilon/3(2^n)$ so that

$$d(x, y) < \delta \text{ implies } d_n(x, y) < \varepsilon/3.$$

Now, by definition of d_n ,

$$d_n(x, y) + 2\rho^*(p_n, x \wedge_{p_n} y) = \rho^*(p_n, x) + \rho^*(p_n, y).$$

Since ρ^* is monotone on arcs, we have

$$\rho^*(p_n, x \wedge_{p_n} y) \leq \rho^*(p_n, x) < \varepsilon/3.$$

Therefore,

$$\rho^*(x, y) \leq \rho^*(p_n, x) + \rho^*(p_n, y) = d_n(x, y) + 2\rho^*(p_n, x \wedge_{p_n} y) < \varepsilon/3 + 2\varepsilon/3 = \varepsilon.$$

This concludes the proof of the main theorem.

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