PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 2, June 1992

ON SOME SUMMABILITY FACTORS OF INFINITE SERIES

W. T. SULAIMAN

(Communicated by Andrew M. Bruckner)

ABSTRACT. A new theorem concerning some summability factors of infinite series is proved. Other results, some of them known, are also deduced.

1. INTRODUCTION

Let $\sum a_n$ be an infinite series of partial sums s_n. Let σ_n^δ and η_n^δ denote the nth Cesàro mean of order $\delta (\delta > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series $\sum a_n$ is said to be absolutely summable (C, δ) with index k, or simply summable $|C, \delta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\delta - \sigma_{n-1}^\delta|^k < \infty,$$

or equivalently

$$\sum_{n=1}^{\infty} n^{-1} |\eta_n|^k < \infty.$$

Let $\{p_n\}$ be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty \qquad (P_{-1} = p_{-1} = 0).$$

The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |T_n - T_{n-1}|^k < \infty \qquad \text{(Bor [1])},$$

where

$$T_n = P_n^{-1} \sum_{v=0}^{n} p_v s_v.$$

If we take $p_n = 1$, then $|\overline{N}, p_n|_k$ summability is equivalent to $|C, 1|_k$ summability. $|\overline{N}, p_n|_1$ is the same as $|\overline{N}, p_n|$. In general, the two summability methods $|C, \delta|_k$ and $|\overline{N}, p_n|_k$ are not comparable.

Received by the editors February 7, 1990.
1991 *Mathematics Subject Classification*. Primary 40G99.

Here we give the following new definition: Let $\{\varphi_n\}$ be any sequence of positive real constants. The series $\sum a_n$ is said to be summable $|\overline{N}, p_n, \varphi_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |T_n - T_{n-1}|^k < \infty .$$

Clearly $|\overline{N}, p_n, P_n/p_n|_k = |\overline{N}, p_n|_k$, $|\overline{N}, p_n, 1|_1 = |\overline{N}, p_n|$, and $|\overline{N}, 1, n|_k = |C, 1|_k$. The following two results are due to Bor:

Theorem 1. *Let $\{p_n\}$ be a sequence of positive real constants such that as $n \to \infty$*

$$\text{(1.1)} \qquad \begin{aligned} &\text{(i)}\ \ np_n = 0(P_n), \\ &\text{(ii)}\ \ P_n = 0(np_n). \end{aligned}$$

If $\sum a_n$ is summable, $|C, 1|_k$, then it is also summable $|\overline{N}, p_n|_k$, $k \geq 1$.

Theorem 2. *Let $\{p_n\}$ be a sequence of positive real constants such that it satisfies (1.1). If $\sum a_n$ is summable $|\overline{N}, p_n|_k$, then it is also summable $|C, 1|_k$.*

We prove the following:

Theorem 3. *Let $\{p_n\}$, $\{q_n\}$, and $\{\varphi_n\}$ be sequences of positive real constants such that $\{\varphi_n q_n/Q_n\}$ is nonincreasing. Let t_n denote the $(\overline{N}, p_n)$-mean of the series $\sum a_n$. If*

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^k \left(\frac{q_n}{Q_n}\right)^k \varphi_n^{k-1} |\varepsilon_n|^k |\Delta t_{n-1}|^k < \infty ,$$

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |\varepsilon_n|^k |\Delta t_{n-1}|^k < \infty ,$$

and

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^k \varphi_n^{k-1} |\Delta\varepsilon_n|^k |\Delta t_{n-1}|^k < \infty ,$$

then the series $\sum a_n \varepsilon_n$ is summable $|\overline{N}, q_n, \varphi_n|_k$, $k \geq 1$, where $\Delta f_n = f_n - f_{n+1}$ and $Q_n = \sum_{v=0}^{n} q_v \to \infty$ as $n \to \infty$ $(Q_{-1} = q_{-1} = 0)$.

2. Proof of Theorem 3

Let T_n be the $(\overline{N}, q_n)$-mean of the series $\sum a_n \varepsilon_n$. Then we have

$$T_n = \frac{1}{Q_n} \sum_{v=0}^{n} q_v \sum_{r=0}^{v} a_r \varepsilon_r = \frac{1}{Q_n} \sum_{v=0}^{n} (Q_n - Q_{v-1}) a_v \varepsilon_v .$$

Hence

$$T_n - T_{n-1} = \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} a_v \varepsilon_v .$$

Abel's transformation gives

$$\begin{aligned} T_n - T_{n-1} &= \frac{q_n}{Q_n Q_{n-1}} \left\{ \sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} P_{r-1} a_r \right) \Delta(P_{v-1}^{-1} Q_{v-1} \varepsilon_v) \right. \\ &\qquad \left. + \left(\sum_{r=1}^{n} P_{r-1} a_r \right) P_{n-1}^{-1} Q_{n-1} \varepsilon_n \right\} \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left\{ -Q_{v-1} \varepsilon_v \Delta t_{v-1} + \frac{P_{v-1}}{p_v} q_v \varepsilon_v \Delta t_{v-1} - \frac{P_{v-1}}{p_v} Q_v \Delta \varepsilon_v \Delta t_{v-1} \right\} \\ &\quad - \left(\frac{P_n}{p_n} \right) \left(\frac{q_n}{Q_n} \right) \varepsilon_n \Delta t_{n-1} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \quad \text{say}. \end{aligned}$$

To prove the theorem, it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=1}^{\infty} \varphi_n^{k-1} |T_{n,r}|^k < \infty, \qquad r = 1, 2, 3, 4.$$

Applying Hölder's inequality,

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{k-1} |T_{n,1}|^k &= \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k \left| \sum_{v=1}^{n-1} -\frac{Q_{v-1}}{q_v} q_v \varepsilon_v \Delta t_{v-1} \right|^k \\ &\le \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\frac{q_n}{Q_n} \right)^k \frac{1}{Q_{n-1}} \\ &\quad \times \sum_{v=1}^{n-1} \left(\frac{Q_v}{q_v} \right)^k q_v |\varepsilon_v|^k |\Delta t_{v-1}|^k \left\{ \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right\}^{k-1} \\ &\le 0(1) \sum_{v=1}^{m} \left(\frac{Q_v}{q_v} \right)^k q_v |\varepsilon_v|^k |\Delta t_{v-1}|^k \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n q_n}{Q_n} \right) \frac{q_n}{Q_n Q_{n-1}} \\ &\le 0(1) \sum_{v=1}^{m} \varphi_v^{k-1} Q_v |\varepsilon_v|^k |\Delta t_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{q_n}{Q_n Q_{n-1}} \\ &\le 0(1) \sum_{v=1}^{m} \varphi_v^{k-1} |\varepsilon_v|^k |\Delta t_{v-1}|^k , \end{aligned}$$

$$\sum_{n=2}^{m+1} \varphi_n^{k-1} |T_{n,2}|^k = \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}}\right)^k \left| \sum_{v=1}^{n-1} -\frac{P_{v-1}}{p_v} q_v \varepsilon_v \Delta t_{v-1} \right|^k$$

$$\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k q_v |\varepsilon_v|^k |\Delta t_{v-1}|^k \left\{ \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right\}^{k-1}$$

$$\leq 0(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k q_v |\varepsilon_v|^k |\Delta t_{v-1}|^k \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n q_n}{Q_n}\right)^{k-1} \frac{q_n}{Q_n Q_{n-1}}$$

$$\leq 0(1) \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{P_v}{p_v}\right)^k \left(\frac{q_v}{Q_v}\right)^{k-1} q_v |\varepsilon_v|^k |\Delta t_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{q_n}{Q_n Q_{n-1}}$$

$$\leq 0(1) \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{P_v}{p_v}\right)^k \left(\frac{q_v}{Q_v}\right)^k |\varepsilon_v|^k |\Delta t_{v-1}|^k,$$

$$\sum_{n=2}^{m+1} \varphi_n^{k-1} |T_{n,3}|^k = \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}}\right)^k \left| \sum_{v=1}^{n-1} -\frac{P_{v-1}}{p_v} \frac{Q_v}{q_v} q_v \Delta\varepsilon_v \Delta t_{v-1} \right|^k$$

$$\leq \sum_{n=2}^{m+1} \varphi_n^{k-1} \left(\frac{q_n}{Q_n}\right)^k \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{p_v}\right)^k \left(\frac{Q_v}{q_v}\right)^k q_v |\Delta\varepsilon_v|^k |\Delta t_{v-1}|^k$$

$$\times \left\{ \frac{1}{Q_{n-1}} \sum_{v=1}^{n-1} q_v \right\}^{k-1}$$

$$\leq 0(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^k \left(\frac{Q_v}{q_v}\right)^k q_v |\Delta\varepsilon_v|^k |\Delta t_{v-1}|^k \sum_{n=v+1}^{m+1} \left(\frac{\varphi_n q_n}{Q_n}\right)^{k-1}$$

$$\times \frac{q_n}{Q_n Q_{n-1}}$$

$$\leq 0(1) \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{P_v}{p_v}\right)^k Q_v |\Delta\varepsilon_v|^k |\Delta t_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{q_n}{Q_n Q_{n-1}}$$

$$\leq 0(1) \sum_{v=1}^{m} \varphi_v^{k-1} \left(\frac{P_v}{p_v}\right)^k |\Delta\varepsilon_v|^k |\Delta t_{v-1}|^k, \text{ and}$$

$$\sum_{n=1}^{m} \varphi_n^{k-1} |T_{n,4}|^k = \sum_{n=1}^{m} \varphi_n^{k-1} \left| -\left(\frac{P_n}{p_n}\right) \left(\frac{q_n}{Q_n}\right) \varepsilon_n \Delta t_{n-1} \right|^k$$

$$\leq 0(1) \sum_{n=1}^{m} \varphi_n^{k-1} \left(\frac{P_n}{p_n}\right)^k \left(\frac{q_n}{Q_n}\right)^k |\varepsilon_n|^k |\Delta t_{n-1}|^k$$

This completes the proof of the theorem.

3. Applications

Corollary 1. *If*

(3.1)
$$\text{(i) } p_n Q_n = 0(P_n q_n),$$
$$\text{(ii) } P_n q_n = 0(p_n Q_n),$$

then the series $\sum a_n$ *is summable* $|\overline{N}, q_n, \varphi_n|_k$, *whenever it is summable* $|\overline{N}, p_n, \varphi_n|_k$, $k \geq 1$, *and* $\varphi_n = 0(\phi_n)$.

The proof follows from Theorem 3 by putting $\varepsilon_n = 1$.

Corollary 2. *If* (3.1) *is satisfied, then the series* $\sum a_n$ *is summable* $|\overline{N}, q_n|_k$ *whenever it is summable* $|\overline{N}, p_n|_k$, $k \geq 1$.

The proof follows from Corollary 1 by putting $\varphi_n = Q_n/q_n = 0(P_n/p_n)$.

Corollary 3 (Theorems 1 and 2). *If* (1.1) *is satisfied, then the series* $\sum a_n$ *is summable* $|C, 1|_k$ *if and only if it is summable* $|\overline{N}, p_n|_k$, $k \geq 1$.

Proof. ($\Rightarrow$) follows from Corollary 2 by putting $p_n = 1$.
($\Leftarrow$) follows from Corollary 2 by putting $q_n = 1$.

Corollary 4. *If*

$$\text{(i) } \frac{P_n q_n}{p_n Q_n}\varepsilon_n = 0(1),$$

$$\text{(ii) } \varepsilon_n = 0(1),$$

$$\text{(iii) } \frac{P_n}{p_n}\Delta\varepsilon_n = 0(1),$$

then the series $\sum a_n\varepsilon_n$ *is summable* $|\overline{N}, q_n, \varphi_n|_k$, *whenever* $\sum a_n$ *is summable* $|\overline{N}, p_n, \varphi_n|_k$, $k \geq 1$.

The proof follows from Theorem 3.

Corollary 5. *If*

$$\text{(i) } \varepsilon_n = 0(1),$$

$$\text{(ii) } \Delta\varepsilon_n = 0(1/n),$$

then the series $\sum a_n\varepsilon_n$ *is summable* $|C, 1|_k$, *whenever* $\sum a_n$ *is summable* $|C, 1|_k$, $k \geq 1$.

The proof follows from Corollary 4 by putting $p_n = q_n = 1$, $\varphi_n = n$.

Acknowledgments

The author is very grateful to the referee for his interesting comments.

References

1. H. Bor, *On two summability methods*, Math. Proc. Cambridge Philos. Soc. **97** (1985), 147–149.
2. ——, *A note on two summability methods*, Proc. Amer. Math. Soc. **98** (1986), 81–84.

Department of Applied Sciences, College of Technological Studies, P. O. Box 42325, 70654 Kuwait