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THE NUMERICAL RADIUS OF A NILPOTENT OPERATOR ON A HILBERT SPACE

UFFE HAAGERUP AND PIERRE DE LA HARPE

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ABSTRACT. Let T be a bounded linear operator of norm 1 on a Hilbert space H such that $T^n = 0$ for some $n \geq 2$. Then its numerical radius satisfies $w(T) \leq \cos\frac{\pi}{(n+1)}$ and this bound is sharp. Moreover, if there exists a unit vector $\xi \in H$ such that $|\langle T\xi|\xi\rangle| = \cos\frac{\pi}{(n+1)}$, then T has a reducing subspace of dimension n on which T is the usual n-shift. The proofs show that these facts are related to the following result of Fejer : if a trigonometric polynomial $f(\theta) = \sum_{k=-n+1}^{n-1} f_k e^{ik\theta}$ is positive, one has $|f_1| \leq f_0 \cos\frac{\pi}{(n+1)}$; moroever, there is essentially one polynomial for which equality holds.

Let H be a complex Hilbert space, possibly finite dimensional, and let T be a (bounded linear) operator on H. The *numerical radius* of T is

$$w(T) = \operatorname{Sup}\{|\langle T\xi|\xi\rangle| : \xi \in H_1\}$$

where H_1 denotes the unit sphere in H. It is well known that

$$r(T) \leq w(T) \leq \|T\|$$

where $r(T)$ and $\|T\|$ denote respectively the spectral radius and the norm of T. In particular $w(T) = \|T\|$ whenever T is normal. Also, it follows from polarization that

$$w(T) \geq \tfrac{1}{2}\|T\|.$$

For all this and much more, see [Ha1, §18] and [Ha2, Chapter 17]. The subject of this paper is the comparison of $w(T)$ with $\|T\|$ when T is *nilpotent*, namely when $T^n = 0$ for some integer $n \geq 2$. We show in this case that $w(T) \leq \|T\| \cos\frac{\pi}{n+1}$, and equality holds for the n-shift on $\mathbb{C}^n$ (cf. Theorem 1). We also show how this is related to a result of Fejer about *trigonometric polynomials* of the form

$$f(\theta) = \sum_{k=-n+1}^{n-1} f_k e^{ik\theta}$$

with $f_k \in \mathbb{C}$. Such a polynomial is *positive* if $f(\theta) \geq 0$ for all $\theta \in \mathbb{R}$. The computation of the numerical radius for the n-shift implies that $|f_1| \leq f_0 \cos\frac{\pi}{n+1}$ (cf. Theorem 2).

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Theorem 1. *Let T be an operator on H such that $T^n = 0$ for some $n \geq 2$.*

(1) *One has*

$$w(T) \leq \|T\| \cos \frac{\pi}{n+1}$$

and equality holds when T is the n-dimensional shift on the standard hermitian space $\mathbb{C}^n$.

(2) *Suppose morover that $\|T\| = 1$. Suppose that there exists a unit vector $\xi \in H_1$ with $|\langle T\xi|\xi\rangle| = \cos\frac{\pi}{n+1}$. (If $\dim H < \infty$, this holds if and only if $w(T) = \cos\frac{\pi}{n+1}$.) Let V_ξ be the linear span of $\{\xi, T\xi, \dots, T^{n-1}\xi\}$. Then V_ξ is an n-dimensional subspace of H which is reducing for T, and the restriction of T to V_ξ is unitarily equivalent to the n-dimensional shift on $\mathbb{C}^n$.*

The proof uses inequalities on *positive matrices* (see Proposition 3), namely on matrices $\beta = (\beta_{i,j})_{1\leq i,j\leq n} \in M_n(\mathbb{C})$ such that $\langle \beta\xi|\xi\rangle \geq 0$ for all $\xi \in \mathbb{C}^n$; for such a matrix, we write $\beta \geq 0$.

From now on, n is an integer, $n \geq 2$. For the proof of Theorem 1 we will need the following computation. (This is probably standard; see in particular [DH].)

Proposition 1. *Let S be the n-dimensional shift on $\mathbb{C}^n$, given by the matrix*

$$S = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

and set $\xi_0 = \left(\left(\frac{2}{n+1}\right)^{1/2} \sin\frac{k\pi}{n+1}\right)_{1\leq k\leq n} \in \mathbb{C}^n$. *Then*

(1) $\|S\| = 1$,

(2) $w(S) = \cos\frac{\pi}{n+1}$,

(3) *for $\xi \in \mathbb{C}^n$ with $\|\xi\| = 1$, one has $\langle S\xi|\xi\rangle = \cos\frac{\pi}{n+1}$ if and only if $\xi = e^{i\phi}\xi_0$ for some $\phi \in \mathbb{R}$.*

Proof. The first claim is obvious.

For any operator T on any Hilbert space H, one has

$$\begin{aligned} w(T) &= \sup\{|\Re(e^{i\theta}\langle T\xi|\xi\rangle)| : \theta \in \mathbb{R} \text{ and } \xi \in H_1\} \\ &= \tfrac{1}{2}\sup\{|\langle (e^{i\theta}T + e^{-i\theta}T^*)\xi|\xi\rangle| : \theta \in \mathbb{R} \text{ and } \xi \in H_1\} \\ &= \tfrac{1}{2}\sup\{\|e^{i\theta}T + e^{-i\theta}T^*\| : \theta \in \mathbb{R}\}. \end{aligned}$$

Set now

$$A = S + S^* = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

If $D(\theta)$ denotes the unitary diagonal matrix with entries $e^{i\theta}$, $e^{2i\theta}, \dots, e^{ni\theta}$, one has

$$D(\theta)^*(e^{i\theta}S + e^{-i\theta}S^*)D(\theta) = A$$

for all $\theta \in \mathbb{R}$. Consequently $w(S) = \frac{1}{2}\|A\|$.

For $k = 1, \dots, n$, let $\xi_k \in \mathbb{C}^n$ be the vector with coordinates

$$\sin\frac{k\pi}{n+1}, \quad \sin\frac{2k\pi}{n+1}, \quad \dots, \quad \sin\frac{nk\pi}{n+1}.$$

One has

$$A\xi_k = 2\cos\left(\frac{k\pi}{n+1}\right)\xi_k, \qquad k = 1, \dots, n.$$

(This computation appears already in [Lag, Nos. 19–20].) Consequently

$$w(S) = \frac{1}{2}\|A\| = \frac{1}{2}\sup_{1\le k\le n}\left|2\cos\left(\frac{k\pi}{n+1}\right)\right| = \cos\left(\frac{\pi}{n+1}\right),$$

so that Claim (2) holds. Observe that

$$\|\xi_1\|^2 = \sum_{k=1}^{n}\sin^2\left(\frac{k\pi}{n+1}\right) = \frac{n+1}{2},$$

so that $\xi_0 = \xi_1\|\xi_1\|^{-1}$.

Choose now $\xi \in \mathbb{C}^n$ with $\|\xi\| = 1$. As $(\xi_k\|\xi_k\|^{-1})_{1\le k\le n}$ is an orthonormal basis of $\mathbb{C}^n$, there are complex numbers $c_1, \dots, c_n$ with $\sum|c_k|^2 = 1$ and $\xi = \sum c_k\xi_k\|\xi_k\|^{-1}$. Assume that $\langle S\xi|\xi\rangle = \cos\frac{\pi}{n+1}$. Then

$$\cos\frac{\pi}{n+1} = \frac{1}{2}\langle A\xi|\xi\rangle = \sum_{k=1}^{n}|c_k|^2\cos\frac{k\pi}{n+1}.$$

This implies that $|c_1| = 1$ and $c_2 = \cdots = c_n = 0$, so that $\xi = c_1\xi_0$. This shows (3). $\square$

Notation. If $\beta = (\beta_{i,j})_{1\le i,j\le n} \in M_n(\mathbb{C})$ is any given matrix, and if k, l are positive integers with $\max\{k, l\} \ge n+1$, we set $\beta_{k,l} = 0$.

Lemma 1. *Consider an operator T on H such that $T^n = 0$ and $\|T\| \le 1$, and a unit vector $\xi \in H_1$. Define*

$$\beta_{i,j} = \langle T^{i-1}\xi|T^{j-1}\xi\rangle, \qquad i, j \ge 1.$$

Then the matrix $(\beta_{i,j} - \beta_{i+1,j+1})_{1\le i,j\le n}$ is positive.

Proof. Consider complex numbers $c_1, \dots, c_n$ and set

$$\eta = \sum_{k=1}^{n} c_k T^{k-1}\xi \quad .$$

One has $1 - TT^* \ge 0$, because $\|T\| \le 1$, and thus

$$\langle(1 - T^*T)\eta|\eta\rangle = \sum_{i,j=1}^{n} c_i\overline{c_j}(\beta_{i,j} - \beta_{i+1,j+1}) \ge 0.$$

As this holds for any choice of $c_1, \dots, c_n$, the lemma follows. $\square$

Definition. With S and ξ_0 as in Proposition 1, we define a matrix $\alpha = (\alpha_{i,j})_{1\leq i,j\leq n}$ by

$$\alpha_{i,j} = \langle S^{i-1}\xi_0 | S^{j-1}\xi_0\rangle .$$

Observe that $\alpha_{1,1} = \|\xi_0\|^2 = 1$, and that

$$\alpha_{1,2} = \alpha_{2,1} = \cos\frac{\pi}{n+1}$$

by Proposition 1.3. Observe also that $(\alpha_{i,j} - \alpha_{i+1,j+1})_{1\leq i,j\leq n} \geq 0$ by Lemma 1.

Proposition 2. *Let $\beta = (\beta_{i,j})_{1\leq i,j\leq n} \in M_n(\mathbb{C})$ be a matrix such that $\beta_{1,1} = 1$. Assume that the matrix $\gamma \in M_n(\mathbb{C})$ defined by*

$$\gamma_{i,j} = \beta_{i,j} - \beta_{i+1,j+1}$$

is positive. Then

(1) $|\beta_{1,2}| \leq \alpha_{1,2} = \cos\frac{\pi}{(n+1)}$.

(2) *If moreover $\beta_{1,2} = \alpha_{1,2}$, then $\beta = \alpha$.*

Proof. From the definition of γ, one has

$$\beta_{i,j} = \gamma_{i,j} + \gamma_{i+1,j+1} + \cdots + \gamma_{i+k,j+k}$$

with $k = \min\{n-i, n-j\}$. In particular

$$\text{(I)} \qquad \operatorname{trace}(\gamma) = \beta_{1,1} = 1$$

and $\beta_{1,2} = \gamma_{1,2} + \gamma_{2,3} + \cdots + \gamma_{n-1,n}$. As γ is positive, $\gamma_{i,i}$ is real nonnegative for $i = 1, \ldots, n$ and $|\gamma_{i,j}|^2 \leq \gamma_{i,i}\gamma_{j,j}$ for $i, j = 1, \ldots, n$. Consequently

$$\text{(II)} \qquad |\beta_{1,2}| \leq (\gamma_{1,1})^{1/2}(\gamma_{2,2})^{1/2} + \cdots + (\gamma_{n-1,n-1})^{1/2}(\gamma_{n,n})^{1/2} .$$

Let $\eta \in \mathbb{C}^n$ be the vector with coordinates $(\gamma_{1,1})^{1/2}, \ldots, (\gamma_{n,n})^{1/2}$. Then $\|\eta\|^2 = 1$ by (I) and $|\beta_{1,2}| \leq \langle S\eta|\eta\rangle$ by (II), so that Claim (1) follows by Proposition 1.

Recall that a *state* on $M_n(\mathbb{C})$ is a linear form ω on $M_n(\mathbb{C})$ such that $\omega(1) = 1$ and $\omega(T) \geq 0$ whenever $T \geq 0$. Each positive matrix $\delta \in M_n(\mathbb{C})$ with $\operatorname{trace}(\delta) = 1$ defines a state ω_δ by $\omega_\delta(T) = \operatorname{trace}(T\delta)$. In particular, the matrix γ defines a state ω_γ and one has

$$\omega_\gamma(S) = \sum_{i,j=1}^{n} S_{i,j}\gamma_{i,j} = \sum_{j=1}^{n} \gamma_{j,j+1} = \beta_{1,2} .$$

Given vectors $\zeta_1, \zeta_2 \in \mathbb{C}^n$, we denote by $\zeta_1 \otimes \overline{\zeta_2}$ the matrix of the linear endomorphism $\xi \mapsto \langle \xi|\zeta_2\rangle\zeta_1$ of $\mathbb{C}^n$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be the eigenvalues of γ and choose a basis $(\eta_\nu)_{1\leq\nu\leq n}$ of corresponding eigenvectors, so that

$$\gamma = \sum_{\nu=1}^{n} \lambda_\nu \eta_\nu \otimes \overline{\eta_\nu}$$

and

$$S\gamma = \sum_{\nu=1}^{n} \lambda_\nu S\eta_\nu \otimes \overline{\eta_\nu} .$$

The hypothesis for Claim (2) reads now

$$\omega_\gamma(S) = \operatorname{trace}(S\gamma) = \sum_{\nu=1}^{n} \lambda_\nu \langle S\eta_\nu | \eta_\nu \rangle = \cos\frac{\pi}{n+1}.$$

As $\sum_{\nu=1}^{n} \lambda_\nu = \operatorname{trace}(\gamma) = 1$, it follows from Proposition 1 that

$$\lambda_1 = 1, \qquad \lambda_2 = \cdots = \lambda_n = 0,$$

and that $\eta_1 = e^{i\phi}\xi_0$ for some $\phi \in \mathbb{R}$. This can be expressed as

$$\gamma = \xi_0 \otimes \overline{\xi_0}.$$

Denote by $\xi_0^1, \ldots, \xi_0^n$ the coordinates of ξ_0. One has

$$\beta_{i,j} = \xi_0^i \xi_0^j + \xi_0^{i+1}\xi_0^{j+1} + \cdots + \xi_0^{i+k}\xi_0^{j+k}$$

where $k = \min\{n-i, n-j\} = n - \max\{i, j\}$. But $\xi_0^p = \xi_0^{n+1-p}$ for all $p \in \{1, \ldots, n\}$ and

$$\begin{aligned}
\beta_{i,j} &= \xi_0^{n+1-i}\xi_0^{n+1-j} + \xi_0^{n-i}\xi_0^{n-j} + \ldots + \xi_0^{\max\{i,j\}-i+1}\xi_0^{\max\{i,j\}-j+1} \\
&= \sum_{k=\max\{i,j\}}^{n} (S^{i-1}\xi_0)^k (S^{j-1}\xi_0)^k \\
&= \langle S^{i-1}\xi_0 | S^{j-1}\xi_0 \rangle = \alpha_{i,j}
\end{aligned}$$

for all $i, j \in \{1, \ldots, n\}$. □

Lemma 2. *Let T be an operator on H such that $\|T\| = 1$ and $T^n = 0$, and assume that there exists a unit vector $\xi \in H_1$ such that*

$$\langle T\xi | \xi \rangle = \cos\frac{\pi}{n+1}.$$

Define V_ξ to be the subspace of H spanned by $\xi, T\xi, \ldots, T^{n-1}\xi$. Then

(1) *$(T^*)^{k-1}\xi \in V_\xi$ for $k = 1, \ldots, n$.*
(2) *V_ξ is a reducing subspace for T.*

Proof. Lemma 1 and Proposition 2 imply that

$$\text{(III)} \qquad \langle T^{i-1}\xi | T^{j-1}\xi \rangle = \langle S^{i-1}\xi_0 | S^{j-1}\xi_0 \rangle$$

for all $i, j \in \{1, \ldots, n\}$, and in particular V_ξ is isometricly isomorphic to the span of $\xi_0, S\xi_0, \ldots, S^{n-1}\xi_0$, which is all of $\mathbb{C}^n$, so it follows that V_ξ is of dimension n.

Define $\beta \in M_n(\mathbb{C})$ by $\beta_{i,j} = \langle (T^*)^{i-1}\xi | (T^*)^{j-1}\xi \rangle$. Lemma 1 and Proposition 2 imply also that

$$\text{(IV)} \qquad \beta = \alpha$$

and in particular that $\|(T^*)^{i-1}\xi\| = (\alpha_{i,i})^{1/2}$ for $i = 1, \ldots, n$.

Let P be the orthogonal projection from H onto V_ξ. Consider some $k \in \{1, \ldots, n\}$, and let $c_1, \ldots, c_n$ be the (uniquely defined) complex numbers such that

$$P(T^*)^{k-1}\xi = c_1\xi + c_2 T\xi + \cdots + c_n T^{n-1}\xi.$$

As $(T^{\nu-1}\xi)_{1\le\nu\le n}$ is a basis of V_ξ (though not an orthonormal one!) these c_ν's can be written as functions of the numbers

$$\langle (T^*)^{k-1}\xi | T^{i-1}\xi\rangle \quad \text{and} \quad \langle T^{i-1}\xi | T^{j-1}\xi\rangle, \qquad i, j = 1, \dots, n.$$

But (IV) shows that

$$\langle (T^*)^{k-1}\xi | T^{i-1}\xi\rangle = \langle (T^*)^{k+i-2}\xi|\xi\rangle = \langle (S^*)^{k-1}\xi_0 | S^{i-1}\xi_0\rangle$$

for $i = 1, \dots, n$ and (III) reads

$$\langle T^{i-1}\xi | T^{j-1}\xi\rangle = \langle S^{i-1}\xi_0 | S^{j-1}\xi_0\rangle$$

for $i, j = 1, \dots, n$. It follows that the coordinates of $(S^*)^{k-1}\xi_0$ with respect to the non-orthogonal basis $\xi_0, S\xi_0, \dots, S^{n-1}\xi_0$ of $\mathbb{C}^n$ are also $c_1, c_2, \dots, c_n$, i.e.

$$(S^*)^{k-1}\xi_0 = c_1\xi_0 + c_2 S\xi_0 + \cdots + c_n S^{n-1}\xi_0,$$

hence

$$\|P(T^*)^{k-1}\xi\| = \|(S^*)^{k-1}\xi_0\| = (\alpha_{k,k})^{1/2} = \|(T^*)^{k-1}\xi\|.$$

Consequently $(T^*)^{k-1}\xi \in \mathrm{Im}(P) = V_\xi$ and this proves (1).

Define W_ξ to be the linear span of $\xi, T^*\xi, \dots, (T^*)^{n-1}\xi$. One checks as for V_ξ that W_ξ has dimension n, and Claim (1) shows that $W_\xi \subset V_\xi$. Hence $W_\xi = V_\xi$. As V_ξ [respectively W_ξ] is obviously invariant by T [resp. T^*], it follows that V_ξ is invariant by T and T^*. □

Proof of Theorem 1. Let T be an operator on H such that $T^n = 0$. One may assume that $\|T\| = 1$.

(1) Choose a vector $\xi \in H_1$ and define $\beta \in M_n(\mathbb{C})$ by

$$\beta_{i,j} = \langle T^{i-1}\xi | T^{j-1}\xi\rangle.$$

By Lemma 1 and Proposition 2.1, one has $|\beta_{1,2}| \le \cos\frac{\pi}{n+1}$. As this holds for all $\xi \in H_1$, this shows that $w(T) \le \cos\frac{\pi}{n+1}$.

(2) By hypothesis, there exist $\xi \in H_1$ and $\theta \in \mathbb{R}$ such that $\langle T\xi|\xi\rangle = e^{i\theta}\cos\frac{\pi}{n+1}$. Let V_ξ be the linear span of $\xi, T\xi, \dots, T^{n-1}\xi$.

Define $T' = e^{-i\theta}T$. Then V_ξ is a reducing subspace for T' by Lemma 2 and the restriction of T' to V_ξ is unitarily equivalent to the n-dimensional shift S by Proposition 2. Hence V_ξ is reducing for T and the restriction of T to V_ξ is unitarily equivalent to $e^{i\theta}S$. But $D(\theta)e^{i\theta}SD(\theta)^* = S$ if $D(\theta)$ is the unitary diagonal matrix defined in the proof of Proposition 1, so that the proof is complete. □

From Theorem 1, it is easy to deduce the following 1915 result of L. Fejer (see [PS, Problem VI.52]). We are grateful to J. Steinig who brought Fejer's result to our attention.

Theorem 2. *Consider a positive integer $n \ge 2$ and a positive trigonometric polynomial*

$$f(\theta) = \sum_{k=-n+1}^{n-1} f_k e^{ik\theta}$$

of degree at most $n-1$. Assume that $f \ne 0$, so that in particular $f_0 > 0$.

(1) *One has* $|f_1| \leq f_0 \cos\frac{\pi}{n+1}$, *and the constant* $\cos\frac{\pi}{n+1}$ *is the best possible one.*

(2) *Suppose moreover that* $|f_1| = f_0 \cos\frac{\pi}{n+1}$. *Then*

$$f(\theta) = f_0 |g(\theta + \psi)|^2 \quad \textit{for all } \theta \in \mathbb{R}$$

where g *is defined by*

$$g(\phi) = \left(\frac{2}{n+1}\right)^{1/2} \sum_{k=0}^{n-1} \sin\left(\frac{(k+1)\pi}{n+1}\right) e^{ik\phi}, \qquad \phi \in \mathbb{R},$$

and where $\psi = \arg(f_1)$.

Proof. As $f \neq 0$ and $f(\theta) \geq 0$ for all $\theta \in \mathbb{R}$, one has

$$f_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta > 0,$$

and there is no loss of generality if we assume that $f_0 = 1$. A theorem of Fejer and Riesz [Rud, Theorem 8.4.5] shows that there exists a trigonometric polynomial $g(\theta) = \sum_{k=0}^{n-1} g_k e^{ik\theta}$ such that

$$f(\theta) = |g(\theta)|^2 = \sum_{k=-n+1}^{n-1} \left(\sum g_p \overline{g_q}\right) e^{ik\theta}$$

for all $\theta \in \mathbb{R}$, where the second summation is over $p, q \in \{0, 1, \dots, n-1\}$ with $p - q = k$. One has in particular

$$\text{(V)} \qquad \sum_{p=0}^{n-1} |g_p|^2 = 1 \quad \text{and} \quad f_1 = \langle S\eta | \eta \rangle$$

where $\eta \in \mathbb{C}^n$ is the unit vector with coordinates $g_0, \dots, g_{n-1}$. Thus $|f_1| \leq w(S) = \cos\frac{\pi}{n+1}$ by Proposition 1.2.

Suppose now that $|f_1| = \cos\frac{\pi}{n+1}$. Upon replacing f by the polynomial $\theta \mapsto f(\theta - \beta)$ for some $\beta \in \mathbb{R}$, one may assume that $f_1 = \cos\frac{\pi}{n+1}$. It follows from Proposition 1.3 that $\eta = e^{i\phi}\xi_0$ for some $\phi \in \mathbb{R}$. There is no change in (V) if we replace g by $e^{-i\phi}g$, so that we may assume that $\eta = \xi_0$. But then g has precisely the form given in Theorem 2.2. □

Remarks. (a) Let T be as in Theorem 1 and such that $\|T\| = 1$. For each $\zeta \in H_1$ and $\theta \in \mathbb{R}$, set

$$f_{\zeta,k} = \begin{cases} 1 & \text{if } k = 0, \\ \langle T^k \zeta | \zeta \rangle & \text{if } k > 0, \\ \langle \zeta | T^{|k|} \zeta \rangle & \text{if } k < 0, \end{cases}$$

and

$$f_\zeta(\theta) = \sum_{k=-n+1}^{n-1} f_{\zeta,k} e^{ik\theta}.$$

Set also

$$\begin{aligned}X &= 1+\sum_{k\geq 1}((e^{i\theta}T)^k+(e^{-i\theta}T^*)^k)\\ &= (1-e^{i\theta}T)^{-1}+(1-e^{-i\theta}T^*)^{-1}-1\\ &= (1-e^{-i\theta}T^*)^{-1}\{1-e^{-i\theta}T^*+1-e^{i\theta}T\\ &\qquad -(1-e^{-i\theta}T^*)(1-e^{i\theta}T)\}(1-e^{i\theta}T)^{-1}\\ &= (1-e^{-i\theta}T^*)^{-1}\{1-T^*T\}(1-e^{i\theta}T)^{-1}\end{aligned}$$

and

$$\eta = (1-e^{i\theta}T)^{-1}\zeta .$$

Then

$$f_\zeta(\theta) = \langle X\zeta|\zeta\rangle = \langle (1-T^*T)\eta|\eta\rangle \geq 0$$

because $\|T\|=1$. Using now Fejer's Theorem 2.1, we see that

$$|\langle T\zeta|\zeta\rangle| \leq \cos\frac{\pi}{n+1}$$

for all $\zeta\in H_1$. This provides a new proof of the inequality in Theorem 1.1.

(b) Assume now that $n\geq 3$. For each $t\in[0,1]$, set

$$S(t) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0\\ 1 & 0 & 0 & \dots & 0 & 0\\ 0 & t & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & t & 0\end{pmatrix} \in M_n(\mathbb{C})$$

and let $w(t)$ be the numerical radius of $S(t)$. One has $\|S(t)\|=1$ for all $t\in[0,1]$. One has also $w(1)=\cos\frac{\pi}{n+1}$ and $w(0)=\frac{1}{2}$. As $w(t)$ is continuous in T (see [Ha2, Problem 175]), the set of possible numerical radii of nilpotent operators of norm 1 of order at most n is $[\frac{1}{2}, \cos\frac{\pi}{n+1}]$.

On the other hand, let T be an operator of norm 1 on a finite dimensional Hilbert space H (or more generally on a Hilbert space in which there exists a unit vector ξ such that $\|T\xi\|=1$), and assume that $w(T)=\frac{1}{2}$. Then it is known that T has a reducing subspace of dimension 2 on which it is unitarily equivalent to the 2-shift. (See [WC]; we are grateful to A. Sinclair who made us aware of this.)

For $n\geq 3$, let I_n be the set of those positive real numbers t for which there exists an operator T acting on a finite dimensional Hilbert space with the following properties : $t=w(T)$, $\|T\|=1$, $T^n=0$, $T^{n-1}\neq 0$, and T is irreducible (no nontrivial reducing subspace). The two facts just above show that $I_n =]\frac{1}{2}, \cos\frac{\pi}{n+1}]$.

(c) Let $S(t)$ and $w(t)$ be as above, now with $0<t<1$. It is easy to check that $w(t)<\cos\frac{\pi}{n+1}$. Let $(t_\nu)_{\nu\geq 1}$ be a strictly increasing sequence of positive numbers converging to 1. Consider an operator on an infinite dimensional Hilbert space which is the orthogonal sum of the $S(t_\nu)$'s. Such an operator shows that the hypothesis that T attains its numerical radius cannot be omitted in Claim (2) of Theorem 1.

(d) We have illustrated one more appearance of the ubiquitous sequence

$$\left(\cos \frac{\pi}{n+1}\right)_{n \geq 2}.$$

Its recent popularity is due to the work of V. Jones on index for subfactors and all that [Jon], but it appears in many other domains, such as the elementary geometry of regular polygons [Cox, Formula 2.84], graph theory or Fuchsian groups [GHJ], to mention but a few.

The second author wishes to note that the original idea of this paper is due to the first author. The paper itself was written up while we were both enjoying the hospitality of the Mittag-Leffler Institute during the autumn 1988. We are grateful to A. Sinclair and J. Steinig for their useful comments.

REFERENCES

[Cox] H. S. M. Coxeter, *Introduction to geometry*, J. Wiley, New York, 1961.

[DH] K. R. Davidson and J. A. R. Holbrook, *Numerical radii of zero-one matrices*, Michigan Math. J. **35** (1988), 261–267.

[GHJ] F. M. Goodman, P. de la Harpe, and V.F.R. Jones, *Coxeter graphs and towers of algebras*, Springer, New York, 1989.

[Ha1] P. R. Halmos, *Introduction to Hilbert space and the theory of spectral multiplicity*, Chelsea, New York, 1951.

[Ha2] ——, *A Hilbert space problem book*, D. Van Nostrand, Princeton, NJ, 1967.

[Jon] V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.

[Lag] J. L. de Lagrange, *Recherches sur la nature et la propagation du son* (1759), Oeuvres I, pp. 37–148.

[PS] G. Pólya and G. Szegö, *Problems and theorems in analysis*, vol. II, Springer, Berlin, 1976.

[Rud] W. Rudin, *Fourier analysis on groups*, Interscience, New York and London, 1962.

[WC] J. P. Williams and T. Crimmins, *On the numerical radius of a linear operator*, Amer. Math. Monthly **74** (1967), 832–833.

INSTITUT FOR MATEMATIK OG DATALOGI, ODENSE UNIVERSITET, DK-5230 ODENSE M, DENMARK.

SECTION DE MATHÉMATIQUES, UNIVERSITÉ DE GENÈVE, C.P. 240, CH-1211 GENÈVE 24, SWITZERLAND.