PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 2, June 1992

REGULARITY AND σ-ADDITIVITY OF STATES ON QUANTUM LOGICS

MIRKO NAVARA

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In 1977, Béaver and Cook [1] introduced the notion of regularity of states on quantum logics. They presented a generalization of Alexandroff theorem: each regular finitely additive state on a quantum logic is countably additive. Recently, Dvurečenskij, Neubrunn, and Pulmannová [2] observed an incorrectness in the original proof and doubted thus the validity of the result. We construct here a counterexample.

Let us briefly recall necessary notions (see [1] and [2]). (More details on quantum logics can be found in [4] and [5].)

By a (*quantum*) *logic* we mean an orthomodular poset $\mathscr{L}$. It is called σ-*orthocomplete* if each countable orthogonal sequence has a join in $\mathscr{L}$. A *state* on a quantum logic $\mathscr{L}$ is a mapping $\omega : \mathscr{L} \to [0, 1]$ such that (i) $\omega(1) = 1$, (ii) $\omega(a \vee b) = \omega(a) + \omega(b)$ whenever $a \perp b$. A state ω is called *countably additive* if $\omega(\bigvee_{n\in N} a_n) = \sum_{n\in N} \omega(a_n)$ whenever $\{a_n\}_{n\in N}$ is an orthogonal sequence in $\mathscr{L}$ having a join.

Suppose that $\mathscr{P}$ is a subset of $\mathscr{L}$. A state ω on $\mathscr{L}$ is called $\mathscr{P}$-*regular* if for each $\varepsilon > 0$ and each $q \in \mathscr{L}$ there exists $p \in \mathscr{P}$ with $p \leq q$ and $\omega(q \wedge p') < \varepsilon$ (here p' means the complement of p). The set $\mathscr{P}$ is called *finitely coverable* if for each $p \in \mathscr{P}$ and each sequence $\{p_n\}_{n\in N} \subseteq \mathscr{P}$ such that $\bigvee_{n\in N} p'_n$ exists and $p \leq \bigvee_{n\in N} p'_n$ there exists a $k \in N$ satisfying $p \leq \bigvee_{n\leq k} p'_n$.

The theorem of Béaver and Cook we want to consider was formulated as follows: Let $\mathscr{L}$ be a σ-orthocomplete quantum logic and let $\mathscr{P}$ be a finitely coverable subset of $\mathscr{L}$ containing the meet of any sequence in $\mathscr{P}$. Then each $\mathscr{P}$-regular state on $\mathscr{L}$ is countably additive.

In [2] the authors showed that the original proof of Béaver and Cook was applicable only for subadditive states. They also proved that the Béaver-Cook theorem was not valid for quantum logics which were not σ-orthocomplete. They left open the validity of the original result. As we shall show here, there is a counterexample to the Béaver-Cook theorem. The method for obtaining it is however quite different from that of [2].

Let us start with a finite (lattice) logic $\mathscr{S}$ admitting no states (see e.g. [3]). Let us take a trivial logic $\mathscr{T} = \{0_{\mathscr{T}}, 1_{\mathscr{T}}\}$. (We use indices following the

Received by the editors November 29, 1990.
1991 *Mathematics Subject Classification*. Primary 28A60, 81P20.
Key words and phrases. Quantum logic, regular states, countable additivity.

symbols 0, 1, $\leq$, $\wedge$, $\vee$, etc. in order to distinguish which logic we refer to.) As known, the direct product $\mathscr{U} = \mathscr{T} \times \mathscr{S}$ admits exactly one state (see [6]); moreover, this only state attains 1 at all elements greater than or equal to the element $u = (1_{\mathscr{T}}, 0_{\mathscr{S}})$.

Let M be a countable set. For each $C \subseteq M$, let us take a copy, $\mathscr{U}_C$, of the logic $\mathscr{U}$. Let us now denote by u_C the element of $\mathscr{U}_C$ corresponding to $u \in \mathscr{U}$ and let us construct the horizontal sum, $\mathscr{V}$, of the collection $\{\mathscr{U}_C : C \subseteq M\}$ (see [5] for the definition of the horizontal sum). Finally, let us construct our counterexample, $\mathscr{L}$, as a sublogic of the product $\mathscr{W} = \prod_{m \in M} \mathscr{V}$. We define $\mathscr{L}$ as the collection of all $f \in \mathscr{W}$ satisfying the following condition:

> whenever $f(m) \in \mathscr{U}_C \setminus \{0, 1\}$ for some $m \in M$ and $C \subseteq M$, then $m \in C$ and f is constant on C.

First, let us prove that $\mathscr{L}$ is a logic. Trivially, $\mathscr{L}$ is closed under the formation of orthocomplements in $\mathscr{W}$. It remains to be shown that $\mathscr{L}$ is closed under the formation of countable orthogonal joins in $\mathscr{W}$. Let $\{f_n\}_{n \in N}$ be a sequence of mutually orthogonal elements of $\mathscr{L}$ and let f be its join in $\mathscr{W}$. Suppose that $f(m) \in \mathscr{U}_C \setminus \{0, 1\}$ for some $m \in M$ and $C \subseteq M$. All elements $f_n(m)$, $n \in N$, satisfy $f_n(m) \leq_{\mathscr{V}} f(m)$, so they have to belong to $\mathscr{U}_C$. We see therefore that $m \in C$. All f_n with $f_n(m) \neq 0$ must be constant on C. There is some f_i satisfying $f_i(m) \neq 0$. For all f_j such that $f_j(m) = 0$, the relation $f_i \perp f_j$ implies that f_j attains values from $\mathscr{U}_C$ on the whole C. Hence, f_j is constant (equal to zero) on C. Thus, all f_n as well as f are constant on C and, moreover, $f \in \mathscr{L}$. We have proved that $\mathscr{L}$ is a logic.

Notice also that $\mathscr{L}$ is a lattice (this was not required in [1] but it might be important in another context), though the lattice operations in $\mathscr{L}$ do not coincide with those of the lattice $\mathscr{W}$.

For each $C \subseteq M$ and $v \in \mathscr{U}_C$, let us denote by $v \mid C$ the element of $\mathscr{L}$ which attains the value v on C and which vanishes on $M \setminus C$.

Let us now define $\mathscr{P} \subseteq \mathscr{L}$ as the collection containing $0_{\mathscr{W}}$ and all elements of the form $\bigvee_{C \in \mathscr{F}} u_C \mid C$, where $\mathscr{F}$ is a finite collection of mutually disjoint subsets of M. Each element of $\mathscr{P}$ is of a finite length. One can check easily that $\mathscr{P}$ is closed under the formation of meets and that $\mathscr{P}$ is also finitely-coverable.

Finally, let us consider the state space of $\mathscr{L}$. Let ω be a state on $\mathscr{L}$. For each nonempty $C \subseteq M$, the interval logic $\{f \in \mathscr{L} : f \leq 1 \mid C\}$ contains a sublogic $\{v \mid C : v \in \mathscr{U}_C\}$ admitting exactly one state. We obtain that

$$\begin{aligned} \omega(v \mid C) &= \omega(u_C \mid C) \quad \text{for } u_C \leq v, \\ \omega(v \mid C) &= 0 \quad \text{otherwise.} \end{aligned}$$

The logic $\mathscr{L}$ contains only elements of the form $\bigvee_{C \in \mathscr{S}} v_c \mid C$, where $\mathscr{S}$ is a sequence of mutually disjoint subsets of M and $v_C \in \mathscr{U}_C$ for all $C \in \mathscr{S}$. Each state on the sublogic $\{1 \mid C : C \subseteq M\}$ of $\mathscr{L}$ extends uniquely to $\mathscr{L}$. Thus, all states on $\mathscr{L}$ are $\mathscr{P}$-regular but some of them are not σ-additive. The proof is complete.

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DEPARTMENT OF MATHEMATICS, FACULTY OF ELECTRICAL ENGINEERING, TECHNICAL UNIVERSITY OF PRAGUE, 166 27 PRAGUE 6, CZECHOSLOVAKIA