PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 2, June 1992

A GALOIS TYPE THEOREM IN VON NEUMANN ALGEBRAS

HISASHI CHODA

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We shall give a simple proof for a Galois type theorem: Let α be a dual free action of a discrete group G on a factor M. If an automorphism θ of M leaves the fixed point algebra M^α pointwise invariant then there exists a $g_0 \in G$ with $\theta = \alpha_{g_0}$.

Since the Galois theory for von Neumann algebras was initiated by M. Nakamura and Z. Takeda in [8], the theory has been developed by several authors ([1, 4] and see [9] for other references). Recently, Y. Katayama and M. Takesaki [6] establish the asymptotic Galois correspondence for discrete amenable group action on AFD factors. They prove their main theorm by reducing it to the following:

Theorem A [6, Theorem 3.1]. *Let α be a dominant free action of a discrete group G on a properly infinite factor M and M^α the fixed point algebra of M under α. If an automorphism θ of M leaves M^α pointwise invariant, then there exists $g_0 \in G$ with $\theta = \alpha_{g_0}$.*

Ikeshoji [5] shows a theorem of the above type for locally compact group actions on von Neumann algebras. Also, as a theory of this type for discrete groups, in [2], we decided an automorphism fixing the fixed point algebra of a discrete automorphism group.

Let M be a von Neumann algebra, G a countable discrete group, α an action of G on M, and M^α the fixed point algebra of M under the action α. We consider a Galois type theorem under the following conditions:

($*$) There exists a faithful normal expectation ε of $(M^\alpha)'$ onto M', where C' is the commutant of an algebra C.

($**$) There exists a group H of automorphisms of M with the following properties (1) and (2):

(1) H is ergodic on the center of M.

(2) $\alpha_g h = h\alpha_g$ for all $g \in G$, $h \in H$.

We have proved a Galois type theorem:

Theorem B [2, Corollary 3]. *Let M, G, α be as above. Suppose that the conditions $(*)$ and $(**)$ hold and that the action α is free. If an automorphism*

Received by the editors November 27, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L55.
Key words and phrases. Galois theory, von Neumann algebra, factor, action, expectation.

θ of M satisfies the following conditions (1) *and* (2):

(1) $\theta h = h\theta$ *for all* $h \in H$,

(2) $\theta(x) = x$ *for all* $x \in M^\alpha$,

then there exists a $g_0 \in G$ *with* $\theta = \alpha_{g_0}$.

In this paper, we shall show that Theorem A is obtained as an application of Theorem B.

At first, we shall show a lemma:

Lemma. *If α is a dual action of a discrete group G on M then there exists a faithful normal expectation ε of $(M^\alpha)'$ onto M'.*

Proof. Since the action α is dual, by [9, Theorem II.2.4, p. 28], there exists a strictly wandering projection $p \in M$ for α, i.e., $\{\alpha_t(p)\,;\, t \in G\}$ is a partition of the identity such that $\alpha_t(p)\alpha_s(p) = 0$ for $t \neq s$. We can complete the proof of the lemma in the same method as [3, Proof of Corollary 3] or [7, Example] (also cf. [4]). But we give a proof of the rest for the completeness. Put

$$\varepsilon(x) = \sum_{t\in G} \alpha_t(p)x\alpha_t(p) \qquad (x \in (M^\alpha)').$$

Then ε is a faithful normal positive linear mapping. We may assume that M acts on a Hilbert space H and that there exists a unitary representation u$(\cdot)$ of G into H with $\alpha_t = \mathrm{Ad}u(t)$ for all $t \in G$, where, for unitary u preserving M invariant, Adu is an automorphism of M induced by u. In this situation, the von Neumann algebra $(M^\alpha)'$ is generated by $\{u(t)\,;\, t \in G\} \cup M'$.

For any $m \in M'$ and $s \in G$, we have

$$\begin{aligned}
\varepsilon(mu(s)) &= \sum_{t\in G} \alpha_t(p)mu(s)\alpha_t(p) \\
&= m\sum_{t\in G} \alpha_t(p)u(s)\alpha_t(p) = m\sum_{t\in G} \alpha_t(p)\alpha_{st}(p)u(s) \\
&= \begin{cases} m, & (s = e), \\ 0, & (s \neq e), \end{cases}
\end{aligned}$$

where e is the unit of G.

Therefore, the map ε is a faithful normal expectation of $(M^\alpha)'$ onto M'.

Theorem. *Let M be a von Neumann algebra, G a discrete group and α a dual free action of G on M.*

Suppose the condition $(**)$ *holds.*

If an automorphism θ of M satisfies the following conditions (1) *and* (2):

(1) $\theta h = h\theta$ *for all* $h \in H$,

(2) $\theta(x) = x$ *for all* $x \in M^\alpha$,

then there exists a $g_0 \in G$ *with* $\theta = \alpha_{g_0}$.

Proof. By the lemma, the condition $(*)$ for this situation holds, i.e., there exists a faithful normal expectation of $(M^\alpha)'$ onto M'. Then, by Theorem B, there exists a $g_0 \in G$ such that $\theta = \alpha_{g_0}$.

As a corollary, we have the factor case, which shows Theorem A .

Corollary. *Let* α *be a dual free action of a discrete group* G *on a factor* M.

If an automorphism θ *of* M *leaves* M^α *pointwise invariant then there exists a* $g_0 \in G$ *with* $\theta = \alpha_{g_0}$.

Proof. Put $\mathrm{H} = \{ \mathrm{id} \}$, where id is the identity automorphism of M. Then H has properties in the condition $(**)$. Hence, the corollary follows from the theorem.

Acknowledgment

The author would like to express his thanks to Prof. Y. Katayama and the referee of this paper for informations on Ikeshoji's paper [5].

References

[1] H. Choda, *A Galois correspondence in a von Neumann algebra*, Tôhoku Math. J. **30** (1978), 491–504.

[2] ——, *An automorphism fixing the fixed point algebra of an automorphism group*, Math. Japon. **24** (1979), 45–51.

[3] M. Choda, *Normal expectations and crossed products of von Neumann algebras*, Proc. Japan Acad. **50** (1974), 738–742.

[4] M. Henle, *Galois theory of W*-algebras*, preprint.

[5] K. Ikeshoji, *A generalization of Roberts-Tannaka duality theorem*, J. Math. Soc. Japan **34** (1982), 55–59.

[6] Y. Katayama and M. Takesaki, *Asymptotic Galois correspondence for discrete amenable actions on AFD factors*, preprint.

[7] N. J. Munch, *Conditional expectations and wandering operators for automorphic group actions*, Math. Japon. **34** (1989), 65–72.

[8] M. Nakamura and Z. Takeda, *A Galois theory for finite factors*, Proc. Japan Acad. **36** (1960), 258–260.

[9] Y. Nakagami and M. Takesaki, *Duality for crossed products of von Neumann algebras*, Lecture Notes in Math., vol. 731, Springer-Verlag, New York and Berlin, 1979.

Department of Mathematics, Osaka Kyoiku University, Tennoji, Osaka, 543, Japan