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STRONG-PRINCIPAL BIMODULES OF NEST ALGEBRAS

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ABSTRACT. We shall show that $B(H)$ can be represented by the strong closure of the linear span of the compounds of a fixed operator in $B(H)$ and the rank one operators, composed only by the vectors of a certain orthonormal basis of H, in a nest algebra, even, under some assumption, in the radical of a nest algebra.

0. NOTATIONS, BASIC RELATIONSHIPS, AND INTRODUCTION

Throughout, the nonzero Hilbert space H under consideration is complex and separable. Subspace means "closed subspace of H" and operator means "bounded linear operator from H into itself." $\subseteq$ is used for "is contained in," while $\subset$ is reserved for "is properly contained in." We write $B(H)$ for the set of all operators and $K(H)$ for the set of all compact operators. The symbol $x \otimes y$ denotes the rank-1 operator $(\cdot, x)y$ for $x, y \in H$. If $S \subseteq H$ and $\mathscr{S} \subseteq \mathbf{B}(H)$, we write $R^1(S)$ for the set of all rank-1 operators composed by vectors in S and $R^1_s\mathscr{S}$ and $R^1\mathscr{S}$ for $R^1(S)\cap\mathscr{S}$ and $R^1(H)\cap\mathscr{S}$ respectively. If M is a subspace, P_M stands for the projection from H onto M. If $\mathscr{S} \subseteq B(H)$, $[\mathscr{S}]^{(s)}$ ($[\mathscr{S}]^{(n)}$) denotes the strong closure (norm closure) of the linear span of $\mathscr{S}$. If $\mathscr{S}, \mathscr{A} \subseteq B(H)$, and if there exists G in $B(H)$ such that $\mathscr{S} = [\mathscr{A} G \mathscr{A}]^{(s)}$ ($\mathscr{S} = [\mathscr{A} G \mathscr{A}]^{(n)}$), then we call $\mathscr{S}$ a strong-principal (norm-principal) bimodule of $\mathscr{A}$ with respect to G.

A nest $\mathscr{N}$ is a family of subspaces totally ordered by inclusion. $\mathscr{N}$ is said to be complete if (i) it contains $\{0\}$ and H and (ii) given any subfamily $\mathscr{N}_0$ of $\mathscr{N}$, the subspaces $\bigwedge\{L: L \in \mathscr{N}_0\}$ and $\bigvee\{L: L \in \mathscr{N}_0\}$ are both members of $\mathscr{N}$. If $N \in \mathscr{N}$, we define $N_- = \bigvee\{L: L \subset N, L \in \mathscr{N}\}$ and $N_+ = \bigwedge\{L: N \subset L, L \in \mathscr{N}\}$. Obviously, if $\mathscr{N}$ is complete, then N_-, $N_+ \in \mathscr{N}$ for all $N \in \mathscr{N}$. Throughout, a nest is complete. The set $\{T: T \in B(H), TN \subseteq N, N \in \mathscr{N}\}$ is called the *nest algebra* (associated with $\mathscr{N}$) and is denoted in this paper by $A\mathscr{N}$. The space $M \ominus N$, with $M, N \in \mathscr{N}$ and $M \supseteq N$, is called an $\mathscr{N}$-interval. A finite $\mathscr{N}$-partition is a finite set $\{E_1, \dots, E_n\}$ of mutually orthogonal $\mathscr{N}$-intervals with $E_1 \oplus \cdots \oplus E_n = H$. The Jacobson radical [5, p. 69] of $A\mathscr{N}$ is denoted by $RA\mathscr{N}$. We can regard the following theorem as another

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definition of the radical, which is actually used in this paper.

Theorem. (Ringrose Criterion, [5, p. 73]). *Let $\mathcal{N}$ be a complete nest. If $T \in A\mathcal{N}$, then $T \in RA\mathcal{N}$ if and only if for each $\varepsilon > 0$ there exists a finite $\mathcal{N}$-partition $\{E_n\}$ such that $\|P_{E_n}TP_{E_n}\| < \varepsilon$ for each n.*

Corollary. [4, p. 21]. *Let $\mathcal{N}$ be a complete nest. Then*

$$RA\mathcal{N} = [\{P_N T P_{N^\perp} : N \in \mathcal{N},\ T \in B(H)\}]^{(n)}.$$

Definition. (Larson, [3, p.418]). Let $\mathcal{N}$ be a complete nest. The symbol $R^\infty A\mathcal{N}$ is defined to be the class of all operators T in $A\mathcal{N}$ with the property that given $\varepsilon > 0$ there exists a (perhaps infinite) $\mathcal{N}$-partition $\{E_n\}$ with $\|P_{E_n}TP_{E_n}\| < \varepsilon$ for each n.

Obviously, $RA\mathcal{N} \subseteq R^\infty A\mathcal{N}$. Finally, the following operator G will play an important role in this paper. We define

$$G = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty}\alpha_{ij}(\cdot, e_i)\right)e_j,$$

where $\{e_n\}$ is an orthonormal basis of H and $\{\alpha_{ij}\}$ is a family of nonzero complex numbers restrained by $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}|\alpha_{ij}|^2 < +\infty$.

It is proved in [2, p. 393] that the strong closure of all finite rank operators in a nest of algebra $A\mathcal{N}$ is equal to $A\mathcal{N}$. Motivated by [1, p. 1], we shall show that $B(H)$ is the strong-principal bimodule of $R^1_{\{e_n\}}A\mathcal{N}$ with respect to G for a certain orthonormal basis $\{e_n\}$ of H, that is, the representation is possible as long as we substitute the basic units $x \otimes y$ of the finite rank operators in the nest algebra by the new basic units $(e_i \otimes e_j)G(e_k \otimes e_l)$, where $e_i \otimes e_j$ and $e_k \otimes e_l$ are rank-1 operators in the nest algebra and are composed only by the elements of a certain orthonormal basis $\{e_n\}$ of H.

1. $R^1_{\{e_n\}}A\mathcal{N}$

Lemma 1.1. (Ringrose, [5, p. 64]). *Let $\mathcal{N}$ be a complete nest, and let x and y be nonzero vectors in H. Then $x \otimes y \in A\mathcal{N}$ if and only if there is an $N \in \mathcal{N}$ such that $x \in (N_-)^\perp$, $y \in N$.*

The above lemma can be written as

$$R^1A\mathcal{N} = \{x \otimes y : x \in (N_-)^\perp,\ y \in N,\ N \in \mathcal{N}\}.$$

If $\{e_n\}$ is an orthonormal basis in H, we have

$$R^1_{\{e_n\}}A\mathcal{N} = \{e_i \otimes e_j : e_i \in (N_-)^\perp,\ e_j \in N;\ e_i, e_j \in \{e_n\};\ N \in \mathcal{N}\}.$$

Lemma 1.2. *Let $\mathcal{N}$ be a complete nest. Then there exists an orthonormal basis $\{e_n\}$ of H with the property that for any $e_i \in \{e_n\}$ there exists $e_j, e_k \in \{e_n\}$ such that*

$$e_j \otimes e_i,\ e_i \otimes e_k \in A\mathcal{N}.$$

Proof. (i) $\{0\} \subset \{0\}_+$ and $H_- \subset H$. Observe that $(\{0\}_+)_- = \{0\}$ and $H = \{0\}^\perp = ((\{0\}_+)_-)^\perp$. Let B_1, B_2, and B_3 denote orthonormal bases of $\{0\}_+$, $H_- \ominus \{0\}_+$ and $(H_-)^\perp$ respectively. We define $\{e_n\} = B_1 \cup B_2 \cup B_3$. For any $e_i \in \{e_n\}$, if $e_i \in B_1$, it follows from $e_i \in H = ((\{0\}_+)_-)^\perp$, $e_i \in \{0\}_+$ that e_i

can act as e_j and e_k as well; If $e_i \in B_2$, for any $e_j \in B_3$, it follows from $e_j \in (H_-)^\perp$, $e_i \in H$ that $e_j \otimes e_i \in A\mathcal{N}$, and for any $e_k \in B_1$, it follows from $e_i \in H = ((\{0\}_+)_-)^\perp$, $e_k \in \{0\}_+$ that $e_i \otimes e_k \in A\mathcal{N}$; If $e_i \in B_3$, it follows from $e_i \in (H_-)^\perp$, $e_i \in H$ that e_i can act as e_j and e_k as well.

(ii) $\{0\} = \{0\}_+$ and $H_- = H$. There exists a sequence $\{N_n\}_{n=-\infty}^{+\infty}$ in $\mathcal{N}$ satisfying (i) if $i < j$, then $N_i \subset N_j$, (ii) $\lim_{n\to-\infty} P_{N_n} = 0$ and $\lim_{n\to+\infty} P_{N_n} = I$ all in the strong operator topology [1, p. 2]. Let B_n denote an orthonormal basis of $N_n \ominus N_{n-1}$ for each n. We define $\{e_n\} = \bigcup_{n=-\infty}^{+\infty} B_n$. For any $e_i \in \{e_n\}$, if $e_i \in B_n$, then any vector in $B_m\,(m > n)$ can act as e_j, while any vector in $B_m\,(m < n)$ can act as e_k.

(iii) The proof for the case either $\{0\} \subset \{0\}_+$ and $H_- = H$ or $\{0\} = \{0\}_+$ and $H_- \subset H$ is trivial. □

Lemma 1.3. *Let $\mathcal{N}$ be a complete nest. Then there exists an orthonormal basis $\{e_n\}$ of H with the property that for any $e_i, e_j \in \{e_n\}$ there is a complex number $\alpha(i, j) \neq 0$ such that*

$$\alpha(i, j)(e_i \otimes e_j) \in (R^1_{\{e_n\}}A\mathcal{N})G(R^1_{\{e_n\}}A\mathcal{N}).$$

Proof. Let $\{e_n\}$ be the orthonormal basis of H defined in Lemma 1.2. For any $e_i, e_j \in \{e_n\}$, we can find $e_{i_0}, e_{j_0} \in \{e_n\}$ such that $e_i \otimes e_{i_0}$, $e_{j_0} \otimes e_j \in A\mathcal{N}$.

$$\begin{aligned} e_i \otimes e_j &= \frac{1}{\alpha_{i_0 j_0}}(Ge_{i_0}, e_{j_0})e_i \otimes e_j \\ &= \frac{1}{\alpha_{i_0 j_0}}(e_{j_0} \otimes e_j)(e_i \otimes Ge_{i_0}) = \frac{1}{\alpha_{i_0 j_0}}(e_{j_0} \otimes e_j)G(e_i \otimes e_{i_0}), \end{aligned}$$

$$\alpha_{i_0 j_0} e_i \otimes e_j = (e_{j_0} \otimes e_j)G(e_i \otimes e_{i_0}) \in (R^1_{\{e_n\}}A\mathcal{N})G(R^1_{\{e_n\}}A\mathcal{N}). \quad \square$$

Theorem 1.4. *Let $\mathcal{N}$ be a complete nest. Then there exists an orthonormal basis $\{e_n\}$ of H such that*

$$B(H) = [(R^1_{\{e_n\}}A\mathcal{N})G(R^1_{\{e_n\}}A\mathcal{N})]^{(S)}.$$

Proof. Let $\{e_n\}$ be the orthonormal basis of H defined in Lemma 1.2. We know that

$$B(H) = [R^1(H)]^{(S)}.$$

For any $x, y \in H$, let $x = \sum_{i=1} \alpha_i e_i$, $y = \sum_{j=1} \beta_j e_j$. $\sum_{j=1}^n \sum_{i=1}^n \alpha_i \beta_j e_i \otimes e_j$ converges in the norm operator topology to $x \otimes y$ [6, p. 8], so we have

$$[R^1(H)]^{(S)} = [R^1(\{e_n\})]^{(S)}.$$

By Lemma 1.3, we have

$$[R^1(\{e_n\})]^{(S)} \subseteq [(R^1_{\{e_n\}}A\mathcal{N})G(R^1_{\{e_n\}}A\mathcal{N})]^{(S)} \subseteq B(H).$$

Therefore,

$$B(H) = [(R^1_{\{e_n\}}A\mathcal{N})G(R^1_{\{e_n\}}A\mathcal{N})]^{(S)}. \quad \square$$

Theorem 1.5. *Let $\mathcal{N}$ be a complete nest. Then there exists an orthonormal basis $\{e_n\}$ of H such that*

$$K(H) = [(R^1_{\{e_n\}}A\mathcal{N})G(R^1_{\{e_n\}}A\mathcal{N})]^{(n)}.$$

Proof. Let $\{e_n\}$ be the orthonormal basis of H defined in Lemma 1.2. We know that

$$K(H) = [R^1(H)]^{(n)} \quad [6, \text{p. } 18].$$

By the proof of Theorem 1.4, we have

$$[R^1(H)]^{(n)} = [R^1(\{e_n\})]^{(n)}.$$

By Lemma 1.3, we have

$$[R^1(\{e_n\})]^{(n)} \subseteq [(R^1_{\{e_n\}}A\mathcal{N})G(R^1_{\{e_n\}}A\mathcal{N})]^{(n)} \subseteq K(H)$$

Therefore, $K(H) = [(R^1_{\{e_n\}}A\mathcal{N})G(R^1_{\{e_n\}}A\mathcal{N})]^{(n)}$. □

2. $R^1_{\{e_n\}}RA\mathcal{N}$ AND $R^1_{\{e_n\}}R^\infty A\mathcal{N}$

Lemma 2.1. *Let $\mathcal{N}$ be a complete nest, and let x and y be nonzero vectors in H. Then $x \otimes y \in RA\mathcal{N}(R^\infty A\mathcal{N})$ if and only if there is $N \in \mathcal{N}$ such that $x \in N^\perp$, $y \in N$.*

Proof. Let x and y be nonzero vectors in H. If there is $N \in \mathcal{N}$ such that $x \in N^\perp$, $y \in N$, by the corollary in §0, it follows from

$$\begin{aligned} x \otimes y &= (\cdot, x)y = (\cdot, P_{N^\perp}x)P_N y \\ &= (P_{N^\perp}\cdot, x)P_N y = P_N(x \otimes y)P_{N^\perp} \end{aligned}$$

that $x \otimes y$ is in $RA\mathcal{N}$ $(R^\infty A\mathcal{N})$.

Now let $x \otimes y \in RA\mathcal{N}$ $(R^\infty A\mathcal{N})$. Since $x \otimes y \in A\mathcal{N}$, there exists $N \in \mathcal{N}$ such that $x \in (N_-)^\perp$, $y \in N$, and since $x \otimes y \in RA\mathcal{N}$ $(R^\infty A\mathcal{N})$, given $\varepsilon > 0$ there is a finite (perhaps infinite) $\mathcal{N}$-partition $\{E_n\}$ such that $\|P_{E_n}(x\otimes y)P_{E_n}\| < \varepsilon$ for all n. Observe that because $N \ominus N_-$ is a minimal interval, there exist some $E_n \supseteq N \ominus N_-$, and it follows from

$$\|P_{N\ominus N-}(x \otimes y)P_{N\ominus N-}\| \leq \|P_{E_n}(x \otimes y)P_{E_n}\| < \varepsilon$$

that

$$P_{N\ominus N-}(x \otimes y)P_{N\ominus N-} = 0$$

so

$$(P_{N\ominus N-}\cdot, x)P_{N\ominus N-}y = 0.$$

Assuming that $N \subset N$ and $y \overline{\in} N-$, we have $(P_{N\ominus N-}\cdot, x) = 0$, i.e. , $x \perp N \ominus N_-$. Together with $x \perp N_-$, thus we have $x \in N^\perp$. □

The above lemma can be written as

$$R^1RA\mathcal{N} = R^1R^\infty A\mathcal{N} = \{x \otimes y \colon x \in N^\perp,\ y \in N;\ N \in \mathcal{N}\}.$$

If $\{e_n\}$ is an orthonormal basis of H, we have

$$R^1_{\{e_n\}}RA\mathcal{N} = R^1_{\{e_n\}}R^\infty A\mathcal{N} = \{e_i \otimes e_j \colon e_i \in N^\perp, e_j \in N; e_i, e_j \in \{e_n\}; N \in \mathcal{N}\}.$$

The proofs for the following Lemma 2.2, Lemma 2.3, and Theorem 2.4 are similar to those of case (ii) of Lemma 1.2, Lemma 1.3, and Theorem 1.4 and so are omitted.

Lemma 2.2. *Let $\mathcal{N}$ be a complete nest with $\{0\} = \{0\}_+$ and $H_- = H$. Then there exists an orthonormal basis $\{e_n\}$ of H with the property that for any $e_i \in \{e_n\}$, there exist $e_j, e_k \in \{e_n\}$ such that $e_j \otimes e_i, e_i \otimes e_k \in RA\mathcal{N}$.*

Lemma 2.3. *Let $\mathcal{N}$ be a complete nest with $\{0\} = \{0\}_+$ and $H_- = H$. Then there exists an orthonormal basis $\{e_n\}$ of H with the property that for any $e_i, e_j \in \{e_n\}$, there exists a complex number $\alpha(i, j) \neq 0$ such that*

$$\alpha(i, j)(e_i \otimes e_j) \in (RA\mathcal{N})G(RA\mathcal{N}).$$

Theorem 2.4. *Let $\mathcal{N}$ be a complete nest with $\{0\} = \{0\}_+$ and $H_- = H$. Then there exists an orthonormal basis $\{e_n\}$ of H such that*

$$B(H) = [(R^1_{\{e_n\}}RA\mathcal{N})G(R^1_{\{e_n\}}RA\mathcal{N})]^{(S)}.$$

Theorem 2.5. *Let $\mathcal{N}$ be a complete nest with either $\{0\} \subset \{0\}_+$ or $H_- \subset H$. Then for any $G \in B(H)$, neither*

$$B(H) = [(RA\mathcal{N})G(RA\mathcal{N})]^{(S)}$$

nor

$$K(H) = [(RA\mathcal{N})G(RA\mathcal{N})]^{(n)}$$

is true.

Proof. If $\{0\} \subset \{0\}_+$, then by Lemma 2.1, we have

$$RA\mathcal{N} \subseteq \{f \otimes e : f \in (\{0\}_+)^\perp,\ e \in H\}.$$

For any $f_1, f_2 \in (\{0\}_+)^\perp$ and $e_1, e_2 \in H$,

$$(f_1 \otimes e_1)G(f_2 \otimes e_2)\{0\}_+ = (Ge_2, f_1)(f_2 \otimes e_1)\{0\}_+ = \{0\}.$$

Thus, evidently, for any nonzero vector e in $\{0\}_+$

$$e \otimes e \,\overline{\in}\, [(RA\mathcal{N})G(RA\mathcal{N})]^{(S)}.$$

If $H_- \subset H$, then by Lemma 2.1, we have

$$RA\mathcal{N} \subseteq \{e \otimes g : e \in H,\ g \in H_-\}.$$

For any $e_1, e_2 \in H$ and $g_1, g_2 \in H_-$

$$\operatorname{ran}(e_1 \otimes g_1)G(e_2 \otimes g_2) = \operatorname{ran}(Gg_2, e_1)(e_2 \otimes g_1) \subseteq H_-.$$

Thus, evidently, for any nonzero vector $f \in (H_-)^\perp$,

$$f \otimes f \,\overline{\in}\, [(RA\mathcal{N})G(RA\mathcal{N})]^{(S)}.$$

In view of the compactness of $e \otimes e$ and $f \otimes f$, neither of the expressions in the theorem is valid. □

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