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# RADON-NIKODYM PROPERTY IN SYMMETRIC SPACES OF MEASURABLE OPERATORS

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ABSTRACT. Let $E$ be a rearrangement invariant function space on $(0, \infty)$ with the RNP. Let $(M, \tau)$ be a von Neumann algebra with a faithful normal semifinite trace $\tau$. It is proved that the associated symmetric space $L_E(M, \tau)$ of measurable operators has the RNP.

## 1. INTRODUCTION

The aim of this note is to solve a problem arising in a previous work [10] concerning the Radon–Nikodym property (RNP) in symmetric spaces of measurable operators. For stating our result, we introduce the necessary definitions and notations (for unexplained notions, see [5, 6, 9]). We also refer the reader to [2] for the definition and the elementary properties of the RNP.

Let $(M, \tau)$ be a semifinite von Neumann algebra acting on a Hilbert space $H$, with a faithful normal semifinite trace $\tau$. Let $\overline{M}$ be the space of all measurable operators with respect to $(M, \tau)$ in the sense of [7], equipped with the measure topology defined there. For $a \in \overline{M}$ and $t > 0$ the $t$th $s$-number (singular number) of $a$ is defined by (cf. [3])

$$\mu_t(a) = \inf\{\|ae\| : e \text{ is a projection in } M \text{ with } \tau(1-e) \leq t\}.$$

The function $t \to \mu_t(a)$ on $(0, \infty)$ is denoted by $\mu(a)$. This is a positive nonincreasing function on $(0, \infty)$. Note that $\mu(a) = \mu(a^*) = \mu(|a|)$, $|a|$ being the absolute value of $a$. Recall the following useful formula for $\mu_t(a)$ in terms of the spectral measures (cf. [3])

$$\mu_t(a) = \inf\{s \geq 0 | \lambda_s(a) \leq t\}, \tag{1}$$

where $\lambda_s(a) = \tau(e_{(s,\infty)}(|a|))$, $e_{(s,\infty)}(|a|)$ being the spectral projection of $|a|$ corresponding to the interval $(s, \infty)$ for $s \geq 0$. It follows immediately that

$$\mu_t(a) = 0 \quad \text{for } t \geq \tau(\text{supp}(a)),$$

where $\text{supp}(a)$, the support of $a$, is the smallest projection $e$ in $M$ such that $ae = a$. In particular, if $\tau$ is finite, i.e., $\tau(1) < \infty$, then for any $a \in \overline{M}$

$$\mu_t(a) = 0 \quad \text{for } t \geq \tau(1). \tag{2}$$

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Now let $E$ be a rearrangement invariant (r.i.) function space on $(0, \infty)$ in the sense of [6]. Then we define the symmetric space of measurable operators associated with $E$ and $(M, \tau)$ as follows:

$$L_E(M, \tau) = \{a \in \overline{M} | \mu(a) \in E\}$$

and

$$\|a\|_{L_E(M, \tau)} = \|a\|_E = \|\mu(a)\|_E, \qquad a \in L_E(M, \tau).$$

It is an elementary fact that $(L_E(M, \tau), \|\cdot\|)$ is a Banach space (cf. [10, Lemma 4.1]). Moreover, we have $\|a\|_E = \|a^*\| = \||a|\|_E$ for any $a \in L_E(M, \tau)$. If $E = L_p(0, \infty)$ $(0 < p \leq \infty)$, then $L_E(M, \tau)$, denoted usually by $L_p(M, \tau)$ in this case, is simply the usual noncommutative $L_p$-space associated with a semifinite von Neumann algebra. Note that $L_\infty(M, \tau)$ is just $M$ equipped with the operator norm.

If $\tau$ is finite, by (2) we have $\mu_t(a) = 0$ for $t \in [\tau(1), \infty)$ $(a \in \overline{M})$; so that in this case $E$ can be taken as an r.i. function space on $(0, \tau(1))$.

It would be natural to expect properties of a symmetric space of measurable operators to reflect the properties of the corresponding r.i. function space. Thus we proved in [11] that uniform convexity and uniform PL-convexity pass from $E$ to $L_E(M, \tau)$, and in [10] that the same phenomenon occurs for the analytic Radon–Nikodym property and the uniform $H$-convexity in the sense of [12]. A related problem arising from [10] is to know whether the RNP also passes from $E$ to $L_E(M, \tau)$. We answer this question by the following result.

**Theorem.** *Let $E$ be an r.i. function space on $(0, \infty)$ having the RNP. Then for any semifinite von Neumann algebra $(M, \tau)$, $L_E(M, \tau)$ possesses the RNP.*

It is worthwhile to note that a similar result for unitary ideals is easy (of course, our theorem contains this special case). In that case, $E$ is a symmetric Banach sequence space on $\mathbf{N}$ and $(M, \tau)$ is just $(B(H), \mathrm{tr})$, where $B(H)$ is the space of all bounded operators on a separable Hilbert space $H$ and tr is the usual trace on $B(H)$. Traditionally, $L_E(B(H), \mathrm{tr})$ is denoted by $C_E$, the unitary ideal associated with $E$. If $E$ has the RNP, $E$ fails to contain $c_0$. Then it follows easily that the canonical basis is a boundedly complete basis for $E$ with a basis constant 1. Therefore $E$ is order isometric to the dual of $E'$, $E'$ being the closed subspace of the dual $E^*$ of $E$ generated by all finite sequences (cf. [5]). Now by a well-known result from the theory of unitary ideals (cf. [4, Theorem III.12.2]), we have

$$(C_{E'})^* = C_{(E')^*} = C_E.$$

Note also that $C_E$ is separable because $E$ is separable. Hence $C_E$ is a separable dual, so it has the RNP. This simple reasoning also shows that for symmetric Banach sequence spaces and unitary ideals, the RNP is equivalent to the absence of $c_0$ in these spaces. Therefore in these spaces the RNP and the analytic RNP coincide.

For further results about symmetric spaces of measurable operators the reader is referred to [10].

## 2. Proof

Now we proceed to prove the theorem. Our proof relies heavily on the idea in the proof of the corresponding result for the analytic RNP in [10]. In the following, $E$ always denotes an r.i. function space on $(0, \infty)$ and $(M, \tau)$, a semifinite von Neumann algebra on $H$, with a faithful normal semifinite trace $\tau$.

As in [10], we first consider the finite trace case.

**Lemma 1.** *Suppose* $\tau(1) = 1$. *Let* $F$ *be an order continuous r.i. function space on* $(0, 1)$. *Then*

$$(L_F(M, \tau))^* = L_{F^*}(M, \tau).$$

*Proof.* Note first that the order continuity of $F$ implies that $F^*$ consists only of measurable functions on $(0, 1)$ (cf. [6]). Hence $F^*$ is also an r.i. function space on $(0, 1)$, so $L_{F^*}(M, \tau)$ is well defined.

The inclusion

$$L_{F^*}(M, \tau) \subset (L_F(M, \tau))^* \quad \text{(of norm } \leq 1)$$

is easily seen. Indeed, by [3, Theorem 4.2], for any $b \in L_{F^*}(M, \tau)$ and any $a \in L_F(M, \tau)$, we have

$$\int_0^1 \mu_t(ab)\, dt \leq \int_0^1 \mu_t(a)\mu_t(b)\, dt.$$

Thus $\tau(ab)$ is well defined and

$$|\tau(ab)| \leq \int_0^1 \mu_t(a)\mu_t(b)\, dt.$$

It follows that the linear functional $l\colon a \mapsto \tau(ab)$ defined on $L_F(M, \tau)$ is continuous and of norm $\leq \|\mu(b)\|_{F^*} = \|b\|_{F^*}$.

For the converse inclusion we take $l \in (L_F(M, \tau))^*$. Then we must show that $l$ is defined by an element $b \in L_{F^*}(M, \tau)$ as above and that the norm of $l$ in $(L_F(M, \tau))^*$ is less or equal to $\|b\|_{L_{F^*}(M, \tau)}$. For this we first show that $l$ is in the predual of $M$, that is, $l$ is an element of $L_1(M, \tau)$ (it is well known that $(L_1(M, \tau))^* = L_\infty(M, \tau) = M$).

Because $F$ is an r.i. function space on $(0, 1)$, we have (cf. [6])

$$L_\infty(0, 1) \subset F \subset L_1(0, 1) \quad \text{(inclusions of norm } \leq 1).$$

It follows that

$$M \subset L_F(M, \tau) \subset L_1(M, \tau) \quad \text{(inclusions of norm } \leq 1).$$

Therefore the continuous linear functional $l$ on $L_F(M, \tau)$ also defined a continuous linear functional on $M$, that is, $l \in M^*$. In order to show that $l$ is in the predual of $M$, by a well-known result from the theory of operator algebras (cf. [9, Corollary III.3.11]), it suffices to show the following: For any orthogonal family $\{e_i\}_{i\in I}$ of projections in $M$

$$l\left(\sum_{i\in I} e_i\right) = \sum_{i\in I} l(e_i). \tag{3}$$

Equation (3) immediately follows from the following lemma.

**Lemma 2.** *Let $\{e_i\}_{i\in I}$ be an orthogonal family of projections of $M$. Then $\sum_{i\in I} e_i$ converges in $L_F(M, \tau)$.*

*Proof*. For each finite subset $J$ of $I$, let $h_J = \sum_{i\in J} e_i$, and let $k_J = \sum_{i \notin J} e_i$ (convergence in the strong operator topology). Then the decreasing net $\{k_J : J$ finite subset of $I\}$ converges to 0 in the strong operator topology. Since $\tau$ is normal and finite, the net $\{\tau(k_J)\}$ converges to 0. Since $F$ is order continuous, $\{\|\chi_{[0,\tau(k_J))]}\|_F\} \to 0$, $\chi_w$ being the indicator function of a subset $w$. As $\mu(k_J) = \chi_{[0,\tau(k_J))]}$, $\{k_J\} \to 0$ in $L_F(M, \tau)$. This proves Lemma 2.

*End of the proof of Lemma* 1. Now $l$ is in the predual of $M$; namely, there exists a measurable operator $b \in L_1(M, \tau)$ such that

$$l(a) = \tau(ab), \qquad \forall a \in M. \tag{4}$$

Consequently

$$|\tau(ab)| \le \|l\| \, \|a\|_F, \qquad \forall a \in M. \tag{5}$$

Now by [10, Lemma 4.5], $M$ is dense in $L_F(M, \tau)$; therefore, for any $a \in L_F(M, \tau)$, $\tau(ab)$ is well defined and (4) and (5) hold for $a \in L_F(M, \tau)$. We next show that $b \in L_{F^*}(M, \tau)$ and $\|b\|_{F^*} \le \|l\|$.

Let $b = u|b|$ be the polar decomposition of $b$ and $|b| = \int_0^\infty t\, de_t$ be the spectral decomposition of $|b|$. Let $\tilde{e}_t = e_{\mu_t(b)-0}$ $(t > 0,\ e_{0-0} = 1)$. Then $|b|$ admits the following Schmidt decomposition (cf. [8])

$$|b| = \int_0^1 \mu_t(b)\, d\tilde{e}_t.$$

Let $x$ be a nonincreasing positive function in $F$ such that $x$ is constant in the intervals where $\mu(b)$ is constant. Define

$$a = \left(\int_0^1 x(t)\, d\tilde{e}_t\right) \cdot u^*.$$

Then it is easy to check that $a \in L_F(M, \tau)$ and $\|a\|_F \le \|x\|_F$. We also have

$$\int_0^1 x(t)\mu_t(b)\, dt = \tau\left(\int_0^1 x(t)\, d\tilde{e}_t \cdot \int_0^1 \mu_t(b)\, d\tilde{e}_t\right) = \tau(ab).$$

Then by (5)

$$\int_0^1 x(t)\mu_t(b)\, dt \le \|l\| \, \|a\|_F \le \|l\| \, \|x\|_F.$$

Taking the supremum in the above inequalities over all $x$ satisfying the previous property and $\|x\|_F \le 1$, we deduce that $\mu(b) \in F^*$ and $\|\mu(b)\|_{F^*} \le \|l\|$. Consequently, $b \in L_{F^*}(M, \tau)$ and $\|b\|_{L_{F^*}(M,\tau)} \le \|l\|$. This concludes the proof of Lemma 1.

**Lemma 3.** *Assume that $\tau$ is finite and $E$ has the RNP. Then $L_E(M, \tau)$ has the RNP.*

*Proof*. After a trivial normalization, we can assume $\tau(1) = 1$. Then we can regard $E$ as an r.i. function space on $(0, 1)$. The RNP of $E$ implies that $E$ is not isomorphic to $L_1(0, 1)$ and that $E$ is maximal in the sense of [6].

Consequently, $E = F^*$, where $F$ is an order continuous r.i. function space on $(0, 1)$ (cf. [6]). Then by Lemma 1

$$L_E(M, \tau) = (L_F(M, \tau))^*.$$

If $L_E(M, \tau)$ was separable, then $L_E(M, \tau)$, as a separable dual, would have the RNP. But unfortunately, $L_E(M, \tau)$ is in general nonseparable. Therefore, we must do something else in order to prove the RNP of $L_E(M, \tau)$.

Now let $X$ be a separable closed subspace of $L_E(M, \tau)$. We show that $X$ is isometric to a closed subspace of a separable dual, from which [2, Corollary III.3.5] and Lemma 3 follows.

Let $\{a_n\}_{n\geq 0}$ be a dense sequence in $X$; by [10, Lemma 4.3], $M$ is dense in $L_E(M, \tau)$. Then by approximating $a_n$ by elements in $M$, we may assume $a_n \in M$ for any $n \geq 0$. Now let $\widetilde{M}$ be the von Neumann subalgebra of $M$ generated by 1 and all the $a_n$'s. Let $\tilde{\tau}$ be the restriction of $\tau$ to $\widetilde{M}$. Clearly, $\tilde{\tau}$ is a faithful normal finite trace on $\widetilde{M}$. It is also clear that $L_E(\widetilde{M}, \tilde{\tau})$ is naturally identified with a closed subspace of $L_E(M, \tau)$ and $X$ a closed subspace of $L_E(\widetilde{M}, \tilde{\tau})$. Now by Lemma 1, $L_E(\widetilde{M}, \tilde{\tau})$ is a dual. Furthermore, by [10, Lemma 5.6], $L_E(\widetilde{M}, \tilde{\tau})$ is separable. Thus, $X$ is a closed subspace of the separable dual $L_E(\widetilde{M}, \tilde{\tau})$, proving Lemma 3.

Lemma 3 proves the theorem in the finite trace case. We reduce the general case to the finite trace case by using the semifiniteness of $\tau$. The following argument is similar to the corresponding part in the proof of Theorem 5.1 in [10]. We outline it only. In the following, $E$ is an r.i. function space on $(0, \infty)$ with the RNP.

We use the following characterization of the RNP due to Bukhvalov and Danilevich [1]. Let $X$ be a Banach space. Let $h_\infty(X)$ denote the space of all bounded harmonic $X$-values functions in the unit disc of the complex plane. Then $X$ has the RNP iff every function $f \in h_\infty(X)$ admits almost everywhere radial limits in $X$ on the unit circle $\mathbf{T}$, that is, $\lim_{r\to 1} f(re^{i\theta})$ exists in $X$ almost everywhere on $\mathbf{T}$.

Let $f \in h_\infty(L_E(M, \tau))$. We show that $f$ admit almost everywhere radial limits in $L_E(M, \tau)$ on $\mathbf{T}$, from which the theorem follows. Write

$$f(re^{i\theta}) = \sum_{n\in\mathbf{Z}} a_n r^{|n|} e^{in\theta}, \qquad 0 \leq r < 1,\ 0 \leq \theta \leq 2\pi,$$

where $a_n \in L_E(M, \tau)$ $(n \in \mathbf{Z})$ and $\limsup_{n\to\pm\infty} \|a_n\|_E^{1/n} \leq 1$. By the semifiniteness of $\tau$ we find an orthogonal family $\{e_i\}_{i\in I}$ of projections in $M$ such that $\tau(e_i) < \infty$ for every $i \in I$ and such that

$$1 = \sum_{i\in I} e_i \quad \text{(convergence in the strong operator topology)}.$$

By [10, Lemma 5.7], for every $n \in \mathbf{Z}$, $\{i \in I : \|a_n e_i\|_E \neq 0 \text{ or } \|e_i a_n\|_E \neq 0\}$ is at most countable; so that there exists an at most countable subset $\{e_k\}_{k\geq 0}$ of $\{e_i\}_{i\in I}$ such that $\|e_k a_n\|_E \neq 0$ or $\|a_n e_k\|_E \neq 0$ for some $n \in \mathbf{Z}$. Let $e = \sum_{k\geq 0} e_k$. Then we have $ef(z) = f(z)e = f(z)$ $(z \in D)$. Therefore replacing $M$ by $eMe$ and $\tau$ by its restriction to $eMe$, we can assume $e = 1$. For $j \geq 0$, $k \geq 0$, set $e_{jk} = e_j \vee e_k$ (maximum taken in the lattice of all the projections in $M$). Now let $M_{e_{jk}} = e_{jk}Me_{jk}$ and $\tau_{e_{jk}}$ be the restriction of $\tau$ to $M_{e_{jk}}$ ($j \geq 0$, $k \geq 0$). $\tau_{e_{jk}}$ is a finite trace on $M_{e_{jk}}$. For $j \geq 0$, $k \geq 0$ consider $f_{jk}(z) = e_j f(z) e_k$ ($z \in D$). Regarded as a function with values in $L_E(M_{e_{jk}}, \tau_{e_{jk}})$, $f_{jk} \in h_\infty(L_E(M_{e_{jk}}, \tau_{e_{jk}}))$ and

$$\|f_{jk}(z)\|_{L_E(M_{e_{jk}}, \tau_{e_{jk}})} \leq \|f(z)\|_{L_E(M, \tau)}, \qquad z \in D.$$

By Lemma 3, $L_E(M_{e_{jk}}, \tau_{e_{jk}})$ has the RNP; so in $L_E(M_{e_{jk}}, \tau_{e_{jk}})$

$$\lim_{r \to 1} f_{jk}(re^{i\theta}) = \varphi_{jk}(e^{i\theta}) \quad \text{almost everywhere on } \mathbf{T}.$$

We can evidently extend the boundary function $\varphi_{jk}$ to a function with values in $L_E(M, \tau)$. This new function is still denoted by $\varphi_{jk}$. Then it satisfies

$$e_j \varphi_{jk}(e^{i\theta}) = \varphi_{jk}(e^{i\theta}) e_k = \varphi_{jk}(e^{i\theta}) \quad \text{almost everywhere on } \mathbf{T};$$

furthermore, the above almost everywhere radial limits also exist in $L_E(M, \tau)$. Then by [10, Lemma 5.7], we can show that $\sum_{j \geq 0} \sum_{k \geq 0} \varphi_{jk}(e^{i\theta})$ converges in $L_E(M, \tau)$ almost everywhere on $\mathbf{T}$ (cf. [10] for more details). Let

$$\varphi(e^{i\theta}) = \sum_{j \geq 0} \sum_{k \geq 0} \varphi_{jk}(e^{i\theta}), \qquad 0 \leq \theta \leq 2\pi.$$

Then $\varphi$ is a bounded measurable function on $\mathbf{T}$ with values in $L_E(M, \tau)$ since for any $m \geq 0$, $n \geq 0$

$$\begin{aligned} \operatorname*{ess\,sup}_{\theta \in \mathbf{T}} \left\| \sum_{j=0}^{m} \sum_{k=0}^{n} \varphi_{jk}(e^{i\theta}) \right\|_{L_E(M, \tau)} &\leq \sup_{z \in D} \left\| \sum_{j=0}^{m} \sum_{k=0}^{n} f_{jk}(z) \right\|_{L_E(M, \tau)} \\ &\leq \sup_{z \in D} \|f(z)\|_{L_E(M, \tau)}. \end{aligned}$$

Let

$$F(re^{i\theta}) = \int_0^{2\pi} \varphi(e^{i\eta}) P_r(\theta - \eta) \frac{d\eta}{2\pi} \qquad (0 \leq r < 1,\ 0 \leq \theta \leq 2\pi)$$

be the Poisson integral of $\varphi$ in the unit disc, $P_r$ being the Poisson kernel. By the dominated convergence theorem

$$F(z) = \sum_{j \geq 0} \sum_{k \geq 0} F_{jk}(z), \qquad z \in D,$$

where $F_{jk}$ is the Poisson integral of $\varphi_{jk}$ ($j \geq 0$, $k \geq 0$). Since $\varphi_{jk}$ is the almost everywhere radial limit of $f_{jk}$, $F_{jk} = f_{jk}$; so that

$$F(z) = \sum_{j \geq 0} \sum_{k \geq 0} f_{jk}(z), \qquad z \in D.$$

On the other hand, it is clear that

$$f(z) = \sum_{j \geq 0} \sum_{k \geq 0} f_{jk}(z), \qquad z \in D.$$

Consequently, $F = f$. Hence $f$ is the Poisson integral of $\varphi$. Therefore

$$\lim_{r \to 1} f(re^{i\theta}) = \varphi(e^{i\theta}) \quad \text{in } L_E(M, \tau) \text{ almost everywhere on } \mathbf{T},$$

which completes the proof of the theorem.


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