PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 2, June 1992

FIXED POINTS FOR DISCONTINUOUS QUASIMONOTONE MAPS IN SEQUENCE SPACES

SABINA SCHMIDT

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In [2] Hu gives a fixed point theorem for discontinuous quasimonotone increasing maps in $X = \mathbb{R}^n$. We will answer the question in [2] as to whether this result can be extended to $X = l^p$, $1 \le p \le \infty$.

Let the Banach spaces $X = \mathbb{R}^n$, c_0, and l^p, $1 \le p \le \infty$, be ordered by the cone $K = \{x = (x_i)_{i\in I} \in X : x_i \ge 0\}$, where $I = \{1, \dots, n\}$ or $I = \mathbb{N}$, respectively. Then (X, K) is order complete, i.e., if a subset A of X has an upper bound, then A has a least upper bound, which we denote by $\sup A$.

For $u, v \in X$, $u \le v$, we set $[u, v] = \{z \in X : u \le z \le v\}$. It is well known (theorem of Tarski; see [4]) that for each increasing map $M: [u, v] \to [u, v]$ the points $\overline{x} = \sup\{x \in [u, v] : x \le Mx\}$ and $\underline{x} = \inf\{x \in [u, v] : x \ge Mx\}$ are fixed points of M.

For a function $f = (f_i)_{i\in I}: X \to X$ we define for $x \in X$, in analogy to the notation in [2],

$$D_\pm f_i(x) = \liminf_{t\to 0\pm} \frac{1}{t}(f_i(x + te^i) - f_i(x)),$$

$$D^\pm f_i(x) = \limsup_{t\to 0\pm} \frac{1}{t}(f_i(x + te^i) - f_i(x)),$$

where e^i, $i \in I$, are the elements of X with components $e^i_j = 1$ for $i = j$, $e^i_j = 0$ for $i \ne j$.

Theorem. *Let $u, v \in X$, $u \le v$, and $f = (f_i)_{i\in I}: [u, v] \to X$ be a function with the following properties:*

(1) $\quad f_i(x) \le f_i(y)$ *for $x, y \in [u, v]$ with $x \le y$ and $x_i = y_i$, $i \in I$;*

(2) $\quad \min\{D_- f_i(x), D_+ f_i(x)\} > -\infty$ *for $x \in [u, v]$, $i \in I$;*

(3)
$$u_i \le f_i(x + (u_i - x_i)e^i), \quad v_i \ge f_i(x + (v_i - x_i)e^i) \qquad \textit{for } x \in [u, v],\ i \in I.$$

Then f has a greatest fixed point $\overline{x}$ and a smallest fixed point $\underline{x}$, and

(4) $\quad \overline{x} = \sup\{x \in [u, v] : x \le f(x)\}, \qquad \underline{x} = \inf\{x \in [u, v] : x \ge f(x)\}.$

Received by the editors July 26, 1990 and, in revised form, November 27, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47H10, 47H05.

In case $X = \mathbb{R}^n$ this result is exactly [2, Theorem 2].

Proof. For each $x \in X$, $i \in I$ define

$$g_i^x(t) = f_i(x + (t - x_i)e^i) \quad \text{for } t \in [u_i, v_i].$$

From (2) and (3) we get $\min\{D_- g_i^x(t), D_+ g_i^x(t)\} > -\infty$ for $t \in [u_i, v_i]$ and $u_i \le g_i^x(u_i)$, $v_i \ge g_i^x(v_i)$.

Hence by [2, Corollary 1] the function $g_i^x(t)$ has a greatest fixed point $M_i x$ and a smallest fixed point $m_i x$ and

$$\begin{aligned} M_i x &= \sup\{t \in [u_i, v_i] : t \le g_i^x(t)\}, \\ m_i x &= \inf\{t \in [u_i, v_i] : t \ge g_i^x(t)\}. \end{aligned} \tag{5}$$

Since, by (1), $g_i^x(t) \le g_i^y(t)$ for each pair $x, y \in [u, v]$ with $x \le y$, we conclude that $M_i x \le M_i y$, $m_i x \le m_i y$ for $x, y \in [u, v]$ with $x \le y$. Therefore $Mx = (M_i x)_{i \in I}$ defines an increasing map $M: [u, v] \to [u, v]$.

Let $\overline{x} = \sup\{x \in [u, v] : x \le Mx\}$ be the greatest fixed point of M. Combining the equations $\overline{x}_i = M_i \overline{x}$ and $g_i^{\overline{x}}(M_i \overline{x})$, we get $\overline{x}_i = g_i^{\overline{x}}(\overline{x}_i) = f_i(\overline{x})$ for each $i \in I$. Therefore $\overline{x}$ is also a fixed point of f. Since $x \le f(x)$ implies $x \le Mx$ by (5), we claim $x \le \overline{x}$ and hence $\overline{x} = \sup\{x \in [u, v] : x \le f(x)\}$.

In a similar way we can prove that the smallest fixed point $\underline{x}$ of the increasing map $m: [u, v] \to [u, v]$ is the smallest fixed point of f and that the second equation in (4) is fulfilled.

Corollary. *Let $u, v \in X$, $u \le v$, and $f = (f_i)_{i \in I}: [u, v] \to X$ be a function with the following properties*:

$$f_i(x) \ge f_i(y) \quad \textit{for } x, y \in [u, v] \textit{ with } x \le y \textit{ and } x_i = y_i, \quad i \in I;$$
$$\max\{D^- f_i(x), D^+ f_i(x)\} < \infty \quad \textit{for } x \in [u, v], \quad i \in I;$$
$$u_i \ge f_i(x + (u_i - x_i)e^i), \quad v_i \le f_i(x + (v_i - x_i)e^i) \quad \textit{for } x \in [u, v], \quad i \in I.$$

Then f has a greatest fixed point $\overline{x}$ and a smallest fixed point $\underline{x}$ and

$$\overline{x} = \sup\{x \in [u, v] : x \ge f(x)\}, \qquad \underline{x} = \inf\{x \in [u, v] : x \le f(x)\}.$$

Remarks. (1) In infinite-dimensional Banach spaces X a function $f: X \to X$ is quasimonotone increasing if $x, y \in X$, $x \le y$, $\varphi \in K^*$, $\varphi(x) = \varphi(y)$ implies $\varphi(f(x)) \le \varphi(f(y))$, where $K^* = \{\varphi \in X^* : \varphi(x) \ge 0 \text{ for all } x \in K\}$ (see Volkmann [3]). For $X = \mathbb{R}^n$, c_0, or l^p, $1 \le p < \infty$, condition (1) is the same as quasimonotonicity. In case $X = l^\infty$ condition (1), even together with (2), is weaker than quasimonotonicity. Choose $\varphi \in K^*$ with $\varphi(x) = 0$ for $x \in c_0$ and $\varphi(e) = 1$ for $e = (e_n)_{n \in \mathbb{N}}$ with $e_n = 1$ for all $n \in \mathbb{N}$. For $x = (x_n)_{n \in \mathbb{N}} \in l^\infty$, define

$$f_n(x) = \begin{cases} 1 & \text{for } x_n \le 0, \\ 1 - n x_n & \text{for } 0 \le x_x \le \frac{1}{n}, \\ 0 & \text{for } x_n \le \frac{1}{n}, \end{cases}$$

Then $f = (f_n)_{n \in \mathbb{N}}: l^\infty \to l^\infty$ satisfies (1) and (3), but is not quasimonotone increasing.

(2) In $X = l^\infty$, methods analogous to these used in this paper also lead to existence theorems for ordinary differential equations with quasimonotone right-hand side (see [1]).

REFERENCES

1. A. Chaljub-Simon and P. Volkmann, *Un théorème d'existence et de comparison pour des équations différentielles dans les espaces de fonctions bornées*, C.R. Acad. Sci. Paris Ser. I Math. (to appear).
2. S.-C. Hu, *Fixed points for discontinuous quasi-monotone maps in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. **104** (1988), 1111–1114.
3. P. Volkmann, *Gewöhnliche Differentialgleichungen mit quasimonoton wachsenden Funktionen in topologischen Vektorräumen*, Math. Z. **127** (1972), 157–164.
4. E. Zeidler, *Nonlinear functional analysis and its applications*, vol. I, Springer-Verlag, Berlin and New York, 1986.

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT KARLSRUHE, KAISERSTRASSE 12, D-7500 KARLSRUHE 1, GERMANY