PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 2, June 1992

# AN INVARIANT ON 3-DIMENSIONAL LIE ALGEBRAS

HIROYUKI TASAKI AND MASAAKI UMEHARA

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We construct an extra symmetric bilinear form on a 3-dimensional Lie algebra $\mathfrak{g}$ which induces an invariant $\chi(\mathfrak{g})$ on $\mathfrak{g}$. Moreover it provides a new viewpoint for the classical classification of 3-dimensional Lie algebras

In this note, we shall construct an extra symmetric bilinear form $S$ on a 3-dimensional Lie algebra, which provides new viewpoints for the classical classification of 3-dimensional Lie algebras.

Let $\mathfrak{g}$ be a 3-dimensional Lie algebra with the Lie bracket $[\ ,\ ]$ over a field $k$ of characteristic $\neq 2$. Let $\{e_1, e_2, e_3\}$ be a fixed basis. There is a canonical identification $\bigwedge^2 \mathfrak{g}^* \cong \mathfrak{g}$ by $e_1 = e_2^* \wedge e_3^*$, $e_2 = e_3^* \wedge e_1^*$, and $e_3 = e_1^* \wedge e_2^*$, where $\{e_1^*, e_2^*, e_3^*\}$ is the dual basis of $\mathfrak{g}^*$ with respect to $\{e_1, e_2, e_3\}$. Then the bracket $[\ ,\ ] \in (\bigwedge^2 \mathfrak{g}^*) \otimes \mathfrak{g}$ is considered as an element of $\mathfrak{g} \otimes \mathfrak{g}$ by the identification and induces a bilinear form $L\colon \mathfrak{g}^* \times \mathfrak{g}^* \to k$, which is invariant under the change of a basis up to scalar multiplications in $k$. If we set the structure constants of the bracket $[\ ,\ ]$ by

$$\begin{aligned}
&[e_2, e_3] = a_{11}e_1 + a_{12}e_2 + a_{13}e_3, \\
&[e_3, e_1] = a_{21}e_1 + a_{22}e_2 + a_{23}e_3, \\
&[e_1, e_2] = a_{31}e_1 + a_{32}e_2 + a_{33}e_3,
\end{aligned} \tag{1}$$

then the representation matrix of $L$ with respect to the basis $\{e_1^*, e_2^*, e_3^*\}$ is written by $A = (a_{ij})_{i,j=1,2,3}$. If we change the basis on $\mathfrak{g}$ by a matrix $P = (p_{ij})_{i,j=1,2,3} \in \mathrm{GL}(3, k)$ such that $e'_j = \sum_{i=1}^{3} p_{ij}e_i$, then new structure constants $A'$ are given by $A' = (\det P)P^{-1}A^{\mathrm{t}}P^{-1}$.

Now, we define another bilinear form $S\colon \mathfrak{g} \times \mathfrak{g} \to k$ by

$$S(u, v) = L(u_1^*, v_1^*)L(u_2^*, v_2^*) - L(u_1^*, v_2^*)L(u_2^*, v_1^*) \quad \text{for } u, v \in \mathfrak{g}, \tag{2}$$

where $u = u_1^* \wedge u_2^*$ and $v = v_1^* \wedge v_2^*$ with respect to the identification $\bigwedge^2 \mathfrak{g}^* \cong \mathfrak{g}$. Then it can be easily checked that the representation matrix of $S$ coincides with the cofactor matrix $A^*$ of $A$. Since $A'^* = {}^{\mathrm{t}}PA^*P$, the bilinear form $S$ is determined independently of the choice of a basis. The following lemma is immediately obtained from the Jacobi identity.

Received by the editors October 3, 1990.
1991 *Mathematics Subject Classification*. Primary 17B05; Secondary 17B30.
*Key words and phrases*. 3-dimensional Lie algebra.

**Lemma.** *The bilinear form* $S$ *is symmetric, namely,* ${}^tA^* = A^*$.

It should be remarked that $\mathfrak{g}$ is unimodular if and only if the matrix $A$ is symmetric. When $k$ is algebraically closed, the isomorphism classes of 3-dimensional unimodular Lie algebras are classified by the rank of the matrix $A$.

**Theorem 1.** *Let* $\mathfrak{g}$ *be a 3-dimensional Lie algebra. Then the bilinear form* $S$ *defined by* (2) *is proportional to the Killing form* $F$ *of* $\mathfrak{g}$.

*Proof.* Let $B$ be a representation matrix of the Killing form $F$. By a straightforward calculation, one can obtain the identity $B = \hat{A}^* - 2A^*$, where $\hat{A}^*$ is the cofactor matrix of $\hat{A} = A - {}^tA$. If $\mathfrak{g}$ is unimodular, then $\hat{A}^* = 0$ and $F = -2S$ holds. So we may assume that $\mathfrak{g}$ is not unimodular. Then $\mathfrak{g}$ is solvable and the basis $\{e_1, e_2, e_3\}$ can be chosen such that $a_{3i} = a_{i3} = 0\,(i = 1, 2, 3)$ (see [1, p. 12]). Then one can easily verify that $\hat{A}^* - 2A^*$ is proportional to $A^*$. This proves the theorem.

By the theorem, we can define an invariant $\chi(\mathfrak{g}) \in \mathbb{P} = k \cup (\infty)$ by $F = (\chi(\mathfrak{g}) - 2)S$, unless $F = S = 0$; namely, $\mathfrak{g}$ is neither Heisenberg nor abdelian. There is another exceptional Lie algebra denoted by $\mathfrak{k}$, which is characterized by the property that the matrix $A$ is skew symmetric. One can easily verify that the well-known classification theorem (e.g., [1, p. 13; 2, Lemma 4.10]) of 3-dimensional Lie algebras is rewritten in the following

**Theorem 2.** *Let* $\mathfrak{g}$ *be a 3-dimensional Lie algebra that is neither unimodular nor isomorphic to* $\mathfrak{k}$. *Then there exists a basis* $\{e_1, e_2, e_3\}$ *of* $\mathfrak{g}$ *such that*

$$(3) \qquad [e_3, e_2] = e_1, \qquad [e_3, e_1] = -e_1 + \frac{1}{\chi(\mathfrak{g})} e_2 \quad \textit{and} \quad [e_1, e_2] = 0.$$

## References

1. N. Jacobson, *Lie algebras*, Intersci. Tracts in Pure Appl. Math., vol. 10, John Wiley-Sons, New York, 1962.
2. J. Milnor, *Curvature of left invariant metrics on Lie groups*, Adv. in Math. **21** (1976), 293-329.

Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305, Japan

Department of Mathematics, College of General Education, Osaka University, Toyonaka, Osaka, 560, Japan

*Current address*: Masaaki Umehara: Department of Mathematics, College of General Education, Osaka University, Toyonaka, Osaka, 560 Japan

*E-mail address*: Hiroyuki Tasaki: a906023@sakura.cc.tsukuba.ac.jp