SOME REMARKS OF DROP PROPERTY

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ABSTRACT. Let C be a proper closed convex set. C is said to have the drop property if for any nonempty closed set A disjoint with C, there is $a \in A$ such that $co(a, C) \cap A = \{a\}$. We show that if X contains a noncompact set with the drop property, then X is reflexive. Moreover, we prove that if C is a noncompact closed convex subset of a reflexive Banach space, then C has the drop property if and only if C satisfies the following conditions: (i) the interior of C is nonempty; (ii) C does not have any asymptote, and the boundary of C does not contain any ray; and (iii) every support point x of C is a point of continuity.

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a real Banach space, and let C be a nonempty proper closed convex subset of X. For any $x \notin C$, the *drop* determined by x is the set D(x, C) = co(x, C), the convex hull of the set $\{x\} \cup C$. Daneš [D] proved that if C is a bounded closed subset of X and A is a closed set at positive distance from C, then there exists an $a \in A$ such that $D(a, C) \cap A = \{a\}$. Modifying the assumption, Rolewicz [R1] said a nonempty proper closed set C has the drop property if for every nonempty closed set A disjoint with C, there exists a point $a \in A$ such that $D(a, C) \cap A = \{a\}$. The bounded closed convex sets with the drop property are studied in [K1, K2, M, R1, R2]. In [R1] Rolewicz proved that if the closed unit ball of X has the drop property (in this case, we say X has the drop property), then X is reflexive. Kutzarova [K1] extended this result by showing X is reflexive if X contains a noncompact bounded closed convex set (respectively, a noncompact balanced closed convex set) with the drop property. Recently, Kutzarova and Rolewicz [KR1] showed that X is reflexive if X contains a noncompact closed convex symmetric set with the drop property.

For any subset C of X, the Kuratowski measure of C is the infimum $\alpha(C)$ of those $\varepsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less then ε . It is known that $\alpha(C) = 0$ if and only if C is totally bounded. Let C be a closed convex subset of X. We denote the set of all nonzero linear functionals $f \in X^*$, which are bounded above C by F(C).

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For any $f \in F(C)$, and any $\delta > 0$, the slice $S(f, C, \delta)$ is the set

$$\{x \in C : f(x) \ge M - \delta\},\$$

where $M = \sup\{f(x) : x \in C\}$. A closed convex set C is said to have property (α) if

$$\lim_{\delta \to 0} \alpha(S(f, C, \delta)) = 0$$

for all $f \in F(C)$. It is easy to see that a closed convex set C has property (α) if and only if for any $f \in F(C)$ and $x_n \in S(f, C, \frac{1}{n})$, $\{x_n\}$ contains a convergent subsequence. In [KR1] Kutzarova and Rolewicz proved the following

Theorem A. Let C be any closed convex subset of X.

(i) If C has the drop property, then C has property (α) .

(ii) If C is not compact and if C has the drop property, then C has nonempty interior.

(iii) Suppose X is reflexive. If C has nonempty interior and C has property (α) , then C has the drop property.

(iv) Let C be a closed bounded convex set of a reflexive Banach space. If $int(C) \neq \emptyset$ (where int(C) is the interior of C) and every support point of C is a point of continuity, then C has drop property.

Using Theorem A, they proved that if C_1 and C_2 are any two bounded sets with the drop property, then $C_1 \cap C_2$, $C_1 + C_2$, and $co(C_1, C_2)$ have the drop property. In §2 we show the assumption of boundedness can be removed. Hence, if X contains a noncompact closed convex set with the drop property, then X is reflexive. This gives an answer to a question of D. N. Kutzarova and S. Rolewicz [KR1].

Let C be a closed convex set. C is said to have property (*) if C contains the ray $\{c + \lambda b : \lambda \ge 0\}$ implies for any $x \in X$, there is $\beta > 0$ such that $x + (\beta + \lambda)b \in C$ for every $\lambda \ge 0$. In §2 we prove that if C is a noncompact proper closed convex set of a reflexive Banach space, then C has the drop property if and only if $int(C) \ne \emptyset$, C has property (*), and every support point of C is a point of continuity. This gives an extension of Theorem A(iv).

Recall a space X is said to have the Kadec-Klee property (or property (H)) if on the unit sphere the weakly convergent sequence is convergent in norm (i.e., if $||x_n|| = 1$ and x_n converges weakly to a unit vector x, then x_n converges to x in norm). V. Montesinos [M] proved that X has the drop property if and only if X is reflexive and X has the Kadec-Klee property. Recall that a sequence $\{x_n\}$ is said to be an ϵ -separate sequence for some $\epsilon > 0$ if $sep(x_n) =$ $\inf\{||x_n - x_m|| : n \neq m\} > \epsilon$. A Banach space X is said to have the uniform *Kadec-Klee property* if for every $\epsilon > 0$ there is a $\delta > 0$ such that if x is a weak limit of a norm one ϵ -separate sequence, then $||x|| < 1 - \delta$. A Banach space is said to be *nearly uniformly convex* (NUC) if for every $\epsilon > 0$ there exists a δ , $1 > \delta > 0$, such that for every sequence $\{x_n\} \subseteq B$ with $sep(x_n) > \epsilon$, we have $co(x_n) \cap (1 - \delta)B \neq \emptyset$. It is easy to see that every (NUC) space has the uniform Kadec-Klee property, and every Banach space with the uniform Kadec-Klee property has the Kadec-Klee property. Huff [H] proved that Xis (NUC) if and only if X is reflexive and X has the uniform Kadec-Klee property. Modifying the theorem, Kutzarova and Rolewicz [KR2] said a closed convex set is (NUC) (respectively (NUC')) with respect to a center $c \in C$ if

442

for every $\epsilon > 0$ there exists a δ , $1 > \delta > 0$, such that for every ϵ -separate sequence $\{x_n\} \subseteq C$

$$co(x_n) \cap (1-\delta)(C-c) \neq \emptyset$$

(respectively, $\overline{co}(x_n) \cap (1-\delta)(C-c) \neq \emptyset$).

It is easy to see that if C is (NUC) with respect to $c \in C$, then C is (NUC') with respect to c. Kutzarova and Rolewicz [KR2] proved that if c is an interior point of C, then C is (NUC) with respect to c if and only if C is (NUC') with respect to any c. They asked whether this is still true if c is a boundary point of C. In §3, we show this is true if C has the drop property. We also give an example to show the assumption of the drop property cannot be removed.

2. On the drop property

In [KR1] Kutzarova and Rolewicz asked whether X is reflexive if X contains a noncompact closed convex with the drop property. The following theorem shows the answer is affirmative.

Theorem 1. Let C_1 and C_2 be any two closed convex subsets of X with the drop property. If $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cap C_2$ has the drop property. Hence, if X contains a noncompact closed convex set with the drop property, then X is reflexive.

Proof. Let A be any closed subset of X such that $A \cap (C_1 \cap C_2) = \emptyset$. If $A \cap C_1 = \emptyset$, then there exists $a \in A$ such that $D(a, C_1) \cap A = \{a\}$. This implies $D(a, (C_1 \cap C_2)) \cap A = \{a\}$. So we may assume that $A \cap C_1 \neq \emptyset$. Since $(A \cap C_1) \cap C_2 = \emptyset$ and C_2 has the drop property, there is $a \in A \cap C_1$ such that $D(a, C_2) \cap (A \cap C_1) = \{a\}$. So

$$D(a, (C_2 \cap C_1)) \cap A \subseteq (D(a, (C_2 \cap C_1)) \cap C_1) \cap A = \{a\},\$$

and $C_1 \cap C_2$ has the drop property.

It is easy to see that if X contains a noncompact closed convex set with the drop property, then X contains a noncompact symmetric closed convex set with the drop property. By [KR1, Proposition 4], X is a reflexive space. \Box

Remark 1. Let C be an unbounded closed convex set of a reflexive space. Kutzarova and Rolewicz proved that if S(f, C, 1) is bounded for some $f \in F(C)$, then C contains a ray $\{c + \beta b : \beta > 0\}$. Moreover, if $c' \in C$, C also contains the ray $\{c' + \beta b : \beta > 0\}$.

Let C_1 and C_2 be any two bounded closed convex sets with the drop property. In [KR1] Kutzarova and Rolewicz proved that $\lambda C_1 + \mu C_2$ and $co(C_1, C_2)$ have the drop property. The following theorem shows that the boundedness can be removed.

Theorem 2. Let C_1 and C_2 be any two closed convex sets with the drop property. If $co(C_1, C_2) \neq X$, then

- (i) for any λ , $\mu \neq 0$, $\lambda C_1 + \mu C_2$ is closed, and it has the drop property;
- (ii) $co(C_1, C_2)$ is closed, and it has the drop property.

Proof. We only prove (ii) and leave the proof of (i) to the reader. If C_1 and C_2 are compact, then $co(C_1, C_2)$ is compact. So we may assume that $co(C_1, C_2)$ has an interior point.

PEI-KEE LIN

First, we show $co(C_1, C_2)$ has property (α) . If $f \in F(co(C_1, C_2))$, then $f \in F(C_1) \cap F(C_2)$. Let x (respectively, x') be a point in C_1 (respectively, C_2) such that

$$f(x) = \sup\{f(y) : y \in C_1\}$$

(respectively, $f(x') = \sup\{f(y) : y \in C_2\}$).

One can easily show that

$$S(f, co(C_1, C_2), \delta) = co(x, x') + (S(f, C_1, \delta) - x) + (S(f, C_2, \delta) - x').$$

(Compare with the proof of [KR1, Theorem 9 (iii)].) So

$$\lim_{\delta\to 0} \alpha(S(f, \operatorname{co}(C_1, C_2), \delta) = 0$$

and

$$co(C_1, C_2)$$
 has property (α).

Suppose that $b \in \overline{\operatorname{co}(C_1, C_2)} \setminus \operatorname{co}(C_1, C_2) \neq \emptyset$. By Hahn-Banach Theorem, there is a linear functional f such that $f(b) \ge f(x)$ for all $x \in \operatorname{co}(C_1, C_2)$. Since $b \in \overline{\operatorname{co}(C_1, C_2)}$ there exist $x_n \in C_1$, $x'_n \in C_2$, and $0 \le \beta_n \le 1$ such that

$$\lim_{n\to\infty}\beta_n x_n + (1-\beta_n) x_n' = b.$$

By passing to a subsequece, we may assume that $\{\beta_n\}$ converges to some β , $0 \le \beta \le 1$. It is easy to see that if $\beta \ne 0$ (respectively, $\beta \ne 1$), then

$$\lim_{n \to \infty} f(x_n) = \sup\{f(y) : y \in C_1\}$$

(respectively,
$$\lim_{n \to \infty} f(x'_n) = \sup\{f(y) : y \in C_2\}$$
).

But C_1 and C_2 have the drop property. Hence, if $\beta \neq 0$ (respectively, $\beta \neq 1$), then $\{x_n\}$ (respectively, $\{x'_n\}$) contains a subsequence that converges to some element

$$x \in \{y \in C_1 : f(y) = \sup\{f(z) : z \in C_1\} (= f(b))\}$$

(respectively, $x' \in \{y \in C_2 : f(y) = \sup\{f(z) : z \in C_2\} (= f(b))\}$).

So if $0 < \beta < 1$, then $b = \beta x + (1 - \beta)x' \in co(C_1, C_2)$. On the other hand, if $\beta = 1$ (respectively, $\beta = 0$), then

$$b = \lim_{n \to \infty} (\beta_n x + (1 - \beta_n) x'_n) \in \overline{D(x, C_2)}$$

(respectively, $b = \lim_{n \to \infty} (\beta_n x_n + (1 - \beta_n) x') \in \overline{D(x', C_1)}$).

By Proposition 5 of [KR1], $D(x, C_2)$ and $D(x', C_1)$ are closed sets. So $b \in co(C_1, C_2)$; we get a contradiction. \Box

Lemma 3. Let C be a closed convex set with nonempty interior. If C has property (α) , then C has property (*).

Proof. Suppose it is not true. There exist $c \in C$ and $b, x \in X$ such that $b \neq 0$, $\{c + \lambda b : \lambda \ge 0\} \subseteq C$ but $\{x + \lambda b : \lambda \ge 0\} \cap C$ is not a ray. By the simple convexity argument (see [KR1, Proof of Lemma 2]), the line $\{x + \lambda b : \lambda \in \mathbb{R}\}$

444

is disjoint with C. Since C has at least one interior point, by Hahn-Banach Theorem, there is $f \in X^*$ such that

$$\inf\{f(x+\lambda b): \lambda \in \mathbb{R}\} \ge M = \sup\{f(y): y \in C\}.$$

This implies f(b) = 0, and S(f, C, M - f(c) + 1) contains a ray. We get a contradiction and C must have property (*). \Box

Remark 2. Let C be a closed convex subset of X. A ray $r = \{x + \lambda y : \lambda > 0\}$ is said to be an asymptote if $r \cap C = \emptyset$, and for any $\epsilon > 0$ there is N > 0 such that $\lambda > N$ implies $d(x + \lambda y, C) = \inf\{||x + \lambda y - c|| : c \in C\} < \epsilon$. Suppose C is a closed convex set with nonempty interior. Then C has property (*) if and only if C does not have any asymptote and the boundary of C does not contain any ray. The proof is left to the reader.

Let C be a closed convex set. $c \in C$ is said to be a support point of C if there exists $f \in X^*$, $f \neq 0$, such that $f(c) = \sup\{f(x) : x \in C\}$. A point c in C is said to be a point of continuity if for every sequence $\{x_n\}$ in C, $\{x_n\}$ converges to c weakly implies $\{x_n\}$ converges to c in norm.

Theorem 4. Let C be a noncompact closed convex subset of a reflexive Banach space. Then the following are equivalent.

- (i) C has the drop property;
- (ii) $int(C) \neq \emptyset$ and C has property (α) ;
- (iii) $int(C) \neq \emptyset$, C has property (*), and every support point x of C is a point of continuity.

Proof. By Theorem A and Lemma 3, we only need to show (iii) implies (ii). First, we claim that for each $f \in F(C)$, $S(f, C, \delta)$ is bounded. Suppose it is not true. There exist $f \in F(C)$ and $\{x_n\} \subseteq C$ such that $\lim_{n\to\infty} ||x_n|| = \infty$ and $\lim_{n\to\infty} f(x_n) = M = \sup\{f(x) : x \in C\}$. Let y be any vector in X such that ||y|| < 2 and f(y) = 1.

Case 1. There is a subsequence of $\{x_n/||x_n||\}$ that conveges weakly to a nonzero vector $b \in X$. Then $r = \{x_1 + \lambda b : \lambda \ge 0\} \subseteq C$ and $f(b) = \lim_{n\to\infty} \frac{f(x_n)}{||x_n||} = 0$, but the ray $\{(M+1)y + \lambda b : \lambda > 0\}$ is disjoint with C. We get a contradiction.

Case 2. The $\{x_n/||x_n||\}$ converges weakly to 0. Without loss of generality, we may assume that 0 is on the boundary of C. So $\{x_n/||x_n||\}$ converges to 0 in norm. This is impossible, and we prove our claim.

Let x_n be any point in $S(f, C, \frac{1}{n})$. Since X is reflexive, $\{x_n\}$ contains a weakly convergent subsequence $\{x_{n_k}\}$, say it converges to $y \in C$ weakly. Clearly, $f(y) = \sup\{f(x); x \in C\}$. So y is a support point, and $\{x_{n_k}\}$ converges to y. This implies C has property (α) . \Box

3. NEARLY UNIFORM CONVEXITY

Recall a closed convex set is said to be (NUC ') with a center a if for every $\epsilon > 0$ there exists a δ , $1 > \delta > 0$ such that for every ϵ -separate sequence in C, $\overline{co}(x_n) \cap (a + (1 - \delta)(C - a)) \neq \emptyset$. It is easy to see that if C is (NUC) with respect to an $a \in int(C)$ if and only if C is (NUC ') with respect to a. In [KR2] D. N. Kutzarova and S. Rolewicz asked whether (NUC) and (NUC ') are equivalent. The following theorem shows the answer is affirmative if C has the drop property.

PEI-KEE LIN

Theorem 5. Let C be a closed convex set with the drop property and $c \in C$. Then C is (NUC) with respect to c if (and only if) C is (NUC') with respect to c.

Proof. Since every compact convex set is (NUC), we may assume that the interior of C is nonempty. Let $\{x_n\}$ be an ϵ -seperate sequence in C. If $\{x_n\}$ is not bounded, then $\overline{co}(x_n)$ contains the ray $r = \{x_1 + \lambda b : \lambda \ge 0\}$ for some $b \ne 0$. By Lemma 3, there exists $\beta > 0$ such that $c + 2(x_1 - c + \frac{\beta}{2}b) = c + 2(x_1 - c) + \beta b \in int(C)$. So $x_1 + \frac{\beta}{2}b \in \overline{co}(x_n) \cap int(c + \frac{1}{2}(C - c)) \ne \emptyset$.

If $\{x_n\}$ is bounded, then by passing to a subsequence we may assume (x_n) converges weakly, say it converges to $y \in c + (1 - \delta)(C - c)$ weakly. Since C has the drop property, y is an interior point of C. This implies $y \in int(c + (1 - \frac{\delta}{2})(C - c))$ and we prove the theorem. \Box

Remark 3. The proof of the above theorem shows that if C has the drop property, then C is (NUC) with respect to c if and only if it satisfies the following condition:

(o) for any $\epsilon > 0$, there is δ , $0 < \delta < 1$, such that if x is a weak limit of an ϵ -seperate sequence in C, then $x \in c + (1 - \delta)(C - c)$.

The following example shows the drop property cannot be removed from the above theorem.

Example 1. Let $\{e_n\}$ be the natural basis of ℓ_2 , and let C be the closed convex hull of $\{e_n : n \in \mathbb{N}\}$. Clearly, $0 \in C$. For any $0 < \delta < 1$ and for any $c \in co\{e_n : n \in \mathbb{N}\}$, $(1-\delta)^{-1}c \notin C$. So C is not (NUC) with respect to 0. We claim that if x is a weak limit of an $\epsilon \sqrt{2}$ -sepaeate sequence $\{x_n\} \subseteq C$, then $x \in (1-\epsilon)C$.

By passing to a subsequence and perturbing (x_n) , we may assume that there exists a block squence $\{z_n\}$ such that $x_n = x + z_n$ and $||z_n||_2 \ge \epsilon$. But $||z_n||_1 \ge ||z_n||_2$. We have $x \in (1 - \epsilon)C$. So C is (NUC') with respect to 0.

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