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A FUNCTIONAL ANALYSIS PROOF OF THE EXISTENCE OF HAAR MEASURE ON LOCALLY COMPACT ABELIAN GROUPS

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ABSTRACT. A simple proof of the existence of Haar measure on locally compact abelian groups is given. The proof uses the Markov-Kakutani fixed-point theorem.

It is very well known that every locally compact group has a Haar measure and that the Haar measure is unique up to a positive multiplicative constant. Several different proofs have been given, all of them somewhat difficult. (See [N] for two proofs as well as references to others). In most of these proofs, the existence and uniqueness of Haar measure are established separately. For compact groups, a simple proof of the existence and uniqueness of Haar measure was given by von Neumann [vN], and his proof can be made even simpler by using the Kakutani fixed-point theorem (see [R2]). For locally compact *abelian* groups, uniqueness of Haar measure is easily established (see [R1, p. 2]). The purpose of this short note is to present a simple proof of the *existence* of Haar measure for these groups. The proof will make use of the Markov-Kakutani fixed-point theorem, which we recall below. It is known that this fixed-point theorem can be used to prove that every locally compact abelian group has an invariant mean (see [P, p. 113]). For compact groups an invariant mean and a Haar measure are the same thing, but for noncompact groups this is obviously not the case.

Theorem (Markov-Kakutani). *Let K be a nonempty compact convex subset of a (Hausdorff) topological vector space. Let $\mathscr{F}$ be a commuting family of continuous affine mappings of K into itself. Then there exists a point $p \in K$ such that $Tp = p$ for all $T \in \mathscr{F}$.*

A proof can be found in [C, pp. 155–156]. (There the theorem is stated only for locally convex spaces, but local convexity is not needed in the proof.)

We will also need two lemmas.

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Lemma 1. *Suppose G is a topological group and N is a neighborhood of the identity in G that is symmetric (i.e., $N^{-1} = N$). Then there exists a subset S of G such that for each g in G the set $gN \cdot N$ contains at* least *one element of S and the set gN contains at* most *one element of S.*

Proof. Let $\mathscr{S}$ be the collection of all subsets T of G such that

$$p^{-1}q \notin N \cdot N \quad \text{for all } p, q \in T.$$

By applying Zorn's lemma, we see that $\mathscr{S}$ has a maximal element S. Now if $g \in G$, then there is some s in S such that $g^{-1}s \in N \cdot N$, for otherwise the set $S \cup \{g\}$ would be a member of $\mathscr{S}$ strictly containing S. Moreover, if there were two distinct points s_1, s_2 in S such that both $g^{-1}s_1$ and $g^{-1}s_2$ were in N, then we would have $s_1^{-1}s_2 = s_1^{-1}gg^{-1}s_2 \in N^{-1} \cdot N = N \cdot N$, a contradiction. Thus, there is at most one s in S such that $g^{-1}s \in N$. $\square$

Lemma 2. *Let X be a vector space, and let X^* denote the space of* all *linear functionals on X with the weak*-topology (i.e., the weak topology induced by X). If K is a closed subset of X^* such that for each $x \in X$ the set $\{\Lambda x : \Lambda \in K\}$ is bounded, then K is compact.*

The proof of this lemma is very similar to the proof of the Banach-Alaoglu theorem and is essentially contained in [DS, pp. 423–424]. A more succinct statement of the conclusion is that every closed bounded set in X^* is compact.

Proof of the existence of Haar measure on locally compact abelian groups. Let G be a locally compact abelian group. Let $C_c(G)$ denote the space of compactly supported continuous functions on G, and let $C_c(G)^*$ denote the space of *all* linear functionals on $C_c(G)$ with the weak*-topology (i.e., the weak topology induced by $C_c(G)$). If $f \in C_c(G)$ and $a \in G$, then f_a (the translate of f by a) is defined by $f_a(x) = f(a+x)$. For each a in G, define $T_a : C_c(G)^* \to C_c(G)^*$ by the equation

$$(T_a\Lambda)(f) = \Lambda(f_a) \qquad (\Lambda \in C_c(G)^*, \ f \in C_c(G)).$$

Then each T_a is a continuous linear operator. To establish the existence of Haar measure on G we must simply show that there is a nonzero positive linear functional on $C_c(G)$ that is fixed by every T_a.

Fix a symmetric neighborhood N of the identity in G with compact closure. Let K be the set of positive linear functionals Λ on $C_c(G)$ that satisfy the following two conditions:

(i) $\Lambda(f) \leq 1$ whenever f is a nonnegative function in $C_c(G)$ that is bounded above by 1 and whose support is contained in $a + N$ for some $a \in G$, and

(ii) $\Lambda(f) \geq 1$ whenever f is a nonnegative function in $C_c(G)$ that is equal to 1 on $a + N + N$ for some $a \in G$.

Then K is clearly closed and convex in $C_c(G)^*$. Moreover, by a partition of unity argument every nonnegative function in $C_c(G)$ can be written as a finite sum of nonnegative continuous functions each of which has support in $a + N$ for some $a \in G$. It follows that condition (i) in the definition of K implies that for each function f in $C_c(G)$, the set $\{\Lambda(f) : \Lambda \in K\}$ is bounded. Therefore by Lemma 2, K is compact.

Let S be as in Lemma 1, and note that the functional that consists of a point mass at each point of S (i.e., the functional $f \mapsto \sum_{s \in S} f(s)$) is in K. Thus K is nonempty.

It is clear from the definition of K that each of the operators T_a maps K into itself. Hence, since the operators T_a ($a \in G$) form a commuting family, the Markov-Kakutani fixed-point theorem shows that they have a common fixed-point in K. Since all the elements of K are nonzero positive linear functionals on $C_c(G)$, the proof is complete. □

References

[C] J. B. Conway, *A course in functional analysis*, Springer-Verlag, New York, 1985.

[DS] N. Dunford and J. Schwartz, *Linear operators, Part* I, Interscience Publ., New York, 1958.

[N] L. Nachbin, *The Haar integral*, D. Van Nostrand Co., Princeton, NJ, 1965.

[vN] J. von Neumann, *Zum Haarschen mass in topologischen gruppen*, Comp. Math. **1** (1934), 106–114.

[P] J. P. Pier, *Amenable locally compact groups*, Interscience Publ., New York, 1984.

[R1] W. Rudin, *Fourier analysis on groups*, Interscience Publ., New York, 1962.

[R2] ——, *Functional analysis*, McGraw-Hill, New York, 1973.

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