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ON CHANGING FIXED POINTS AND COINCIDENCES TO ROOTS

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ABSTRACT. The coincidence problem, finding solutions to $f(x) = g(x)$, can sometimes be converted to a root problem, finding solutions to $\sigma(x) = a$. As an application, we show that for any two maps $f, g: M \to M$, $N(f, g) = |L(f, g)|$ where M is a compact connected nilmanifold, $N(f, g)$ and $L(f, g)$ are the Nielsen and Lefschetz numbers, respectively, of f and g. This result in the case where g is the identity is due to D. Anosov.

1. INTRODUCTION

Let G be a compact connected Lie group and K be a closed subgroup. Denote by $M = G/K$ the homogeneous space of right cosets. In [7] Fadell observed that every selfmap $f: M \to M$ has a fixed point if and only if there is a solution to the root problem $\psi(g) = eK$ ($e \in G$ the unit) for every K-map $\psi: G \to M$. Here K acts on G via $k \circ g = gk^{-1}$ and K acts on M via $k \circ gK = kgK$. The root problem is often easier to analyze, solve [2–4, 11] in particular, because root classes of maps into closed orientable manifolds always have the same index. Accordingly, we convert the fixed point and coincidence problems for maps of nilmanifolds into a root problem and generalize to coincidences (Theorem 3.3) Anosov's result [1] that for any selfmap $f: M \to M$ on a compact nilmanifold M, the Nielsen fixed point classes of f have the same index each of which is $0, +1$, or -1.

Throughout H_* and H^* will denote singular homology and cohomology with integer coefficients, respectively.

During the preparation of the manuscript, we learned that Theorem 3.3 was also obtained by Jezierski [9] and McCord [12] using different methods. We thank Bob Brown for bringing [9] to our attention and Chris McCord for his preprint.

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2. LOCAL NIELSEN ROOT THEORY

In this section we introduce a local version of the root theory as in [2] (see also [3, 4, 11]), which we need in §3.

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Let X and Y be oriented n-manifolds and $e \in Y$. Suppose that $f: U \to Y$ is a map defined on an open subset $U \subset X$ such that the set of roots $R_f = \{x \in U | f(x) = e\}$ is compact in U. Define the *root index* of f in U to be the integer

$$\omega(f; U) = \langle f^* \mu_Y, o_{R_f} \rangle$$

where $o_{R_f} \in H_n(U, U - R_f)$ is the fundamental homology class around R_f, $\mu_Y \in H^n(Y, Y - e)$ is the preferred generator, and $f^*: H^n(Y, Y - e) \to H^n(U, U - R_f)$.

Proposition 2.1. (1) (*Homotopy*) *If $F: U \times I \to Y$ is a homotopy such that $K = \bigcup_t R_{F_t}$ is compact in U then*

$$\omega(F_0; U) = \omega(F_1; U).$$

(2) (*Additivity*) *If $U = \bigsqcup_{i=1}^k U_i$ and $R_{f|U_i}$ is compact in U_i for each i, then*

$$\omega(f; U) = \sum_{i=1}^k \omega(f|U_i; U_i).$$

Proof. (i) For each t, $0 \leq t \leq 1$, the inclusion $j: (U, U - K) \hookrightarrow (U, U - R_{F_t})$ induces j_* that takes o_K to $o_{R_{F_t}}$. Since

$$\{F_t\}: (U, U - K) \to (Y, Y - e)$$

is a homotopy, it follows that

$$\langle j^* F_0^* \mu_Y, o_K \rangle = \langle j^* F_1^* \mu_Y, o_K \rangle.$$

Thus,

$$\langle F_0^* \mu_Y, j_* o_K \rangle = \langle F_1^* \mu_Y, j_* o_K \rangle$$

and hence,

$$\omega(F_0; U) = \langle F_0^* \mu_Y, o_{R_{F_0}} \rangle = \langle F_1^* \mu_Y, o_{R_{F_1}} \rangle = \omega(F_1; U).$$

(ii) Since $R_f = \bigsqcup_{i=1}^k R_{f|U_i}$, it follows that

$$\langle f^* \mu_Y, o_{R_f} \rangle = \sum_{i=1}^k \langle (f|U_i)^* \mu_Y, o_{R_f|U_i} \rangle. \quad \square$$

Two roots $x, y \in R_f$ are said to be *Nielsen equivalent* if there is a path $\alpha: [0, 1] \to U$ such that $\alpha(0) = x$, $\alpha(1) = y$, and $f \circ \alpha \sim e$ (rel. end points). It is easy to verify that this is an equivalence relation on R_f and the equivalence classes are called *root classes* of f. Denote by $\Gamma(f, U)$ the set of root classes of f.

Proposition 2.2. *Each root class is open in R_f and hence the set $\Gamma(f, U)$ is finite.*

Proof. The first assertion follows from [11, V.3.3] and the second is then obvious since R_f is compact in U. $\square$

Given a root class $\alpha \in \Gamma(f, U)$, choose an open neighborhood V of α in U such that $V \cap R_f = \alpha$. Then the *root index of the class* α is given by

$$\omega(f; \alpha) := \omega(f|V; V).$$

Note that $\omega(f; \alpha)$ is defined independent of the choice of V. A root class α is said to be *essential* if $\omega(f; \alpha) \neq 0$.

We now prove a local version of [2, Corollary 3] (or [11, V.7.1]).

Theorem 2.3. [1] *Let U be a connected open subset of an oriented n-manifold X. Let $f\colon U \to Y$ be a map where Y is an oriented triangulated n-manifold with $e \in Y$. Suppose that R_f is compact and α is an essential root class of f. Then for any root class $\beta \in \Gamma(f, U)$, $\omega(f; \alpha) = \omega(f; \beta)$.*

Proof. Since R_f is compact in U, there exists a compact connected subset K such that $R_f \subset \operatorname{int} K \subset K \subset U$.

Suppose that $\beta \in \Gamma(f, U) - \{\alpha\}$. Let $x \in \alpha$ and $y \in \beta$. Choose a path C in K with $C(0) = x$, $C(1) = y$ and a regular neighborhood V of the loop $f \circ C$ in Y such that $f^{-1}(V) \subset \operatorname{int} K$. Since Y is a manifold, there exists an isotopy $H\colon Y \times I \to Y$ such that $H_1 = 1_Y$, $H_t(e) = f \circ C(t)$, and $H_t(y) = y$ for all $y \in Y - V$. Define $F\colon U \times I \to Y$ by $F_t = H_t \circ f$. Then

$$\begin{aligned}\{F_t(C(1-t))\} &= \{H_t \circ f \circ C(1-t)\}\\ &= \{H_t \circ H_{1-t}(e)\} \sim \{e\} \quad \text{(rel. end points).}\end{aligned}$$

Thus x and y are Nielsen equivalent as roots of F in $U \times I$. Hence α and β belong to the same root class $\mathcal{N}$ of F for some $\mathcal{N} \in \Gamma(F, U \times I)$. Let $\mathcal{N}_t = p(\mathcal{N} \cap (U \times \{t\}))$ where $p(x, s) = x$, $s \in I$. Note that $\alpha = \mathcal{N}_1$ and $\beta = \mathcal{N}_0$. Since H_t is the identity map outside V, the set of roots R_F of F is compact. Choose an open neighborhood W of $\mathcal{N}$ in $U \times I$ so that $\operatorname{cl}(W)$ does not contain any other roots of F. For any r, $0 \le r \le 1$, $K_r = p(R_F) - W_r$ is compact in U where $W_r = p(W \cap (U \times \{r\}))$. By uniform continuity, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|r - s| < \delta$ implies $d(F_r(x), F_s(x)) < \varepsilon$ for all $x \in K_r$. This implies that F_s has no roots in K_r and hence $\mathcal{N}_s \subset W_r$. Define $\{\Gamma_t^{r,s}\}\colon U \to Y$ by

$$\Gamma^{r,s}(x, t) = F(x, (1-t)r + ts).$$

Then

$$\Gamma_0^{r,s} = F_r, \qquad \Gamma_1^{r,s} = F_s$$

and $\bigcup_t R_{\Gamma_t^{r,s}} \cap W_r$ is compact in W_r. By 2.1(1), we have

$$\omega(F_r; \mathcal{N}_r) = \omega(\Gamma_0^{r,s}|W_r;\, W_r) = \omega(\Gamma_1^{r,s}|W_r;\, W_r) = \omega(F_s; \mathcal{N}_s).$$

By compactness of I, we conclude that

$$\omega(F_0; \beta) = \omega(F_0; \mathcal{N}_0) = \omega(F_1; \mathcal{N}_1) = \omega(F_1; \alpha);$$

therefore,

$$\omega(H_0 \circ f; \beta) = \omega(f; \alpha).$$

Since H_0 is isotopic to $H_1 = 1_Y$ and Y is orientable, $\omega(H_0 \circ f; \beta) = \omega(f; \beta)$ and thus the proof is complete. □

3. Anosov's Theorem for coincidences

Recall that a compact connected nilmanifold M is a homogeneous space of the form G/Γ where G is a connected and simply connected nilpotent Lie group and Γ is a uniform discrete subgroup. For any two maps $f, g\colon M \to M$,

[1] After the authors sent proofs to the editor, Professor Boju Jiang pointed out that Theorem 2.3 is incorrect. The main Theorem 3.3, however, remains valid provided an alternate argument replaces the role that Theorem 2.3 played in Lemma 3.1. This argument will appear in a subsequent correction.

we let $C_{f,g} = \{x \in M | f(x) = g(x)\}$ be the set of coincidences of f and g. As in [4], two coincidences $x, y \in C_{f,g}$ are *Nielsen equivalent* as coincidences if there is a path α in M from x to y such that $f \circ \alpha \sim g \circ \alpha$ (rel. end points). We also have the notion of coincidence index and coincidence classes. The Lefschetz number $L(f, g)$ and the Nielsen number $N(f, g)$ are defined (see [4, 13, 15]). In this section, we will show that $N(f, g) = |L(f, g)|$, which generalizes a result of Anosov [1] in which case g is the identity (see also [8]).

Lemma 3.1. *Let M be a connected compact nilmanifold of* $\dim M \geq 3$. *For any two maps $f_1, f_2 \colon M \to M$, each coincidence class has the same index.*

Proof. Write $M = G/\Gamma$ where G is connected, simply connected, nilpotent and Γ is a uniform discrete subgroup. By a result of Schirmer [14], we may assume without loss of generality that either $C_{f_1, f_2} = \varnothing$ or all the coincidence classes are essential, each of which contains only a single point. If $C_{f_1, f_2} = \varnothing$, the assertion is obvious.

Now let $g_1, \dots, g_k$ be in G such that $C_{f_1, f_2} = \{g_1\Gamma, \dots, g_k\Gamma\}$. We can find Γ-maps $\psi_1, \psi_2 \colon G \to M$ where $\psi_i(g) = g^{-1} f_i(g\Gamma)$, $i = 1, 2$. Let $\tilde{g} = g_j$ for some j, $1 \leq j \leq k$. Thus $\tilde{g}\Gamma \in C_{f_1, f_2}$. Choose lifts $\tilde{\psi}_i \colon G \to G$ of ψ_i, $i = 1, 2$ such that $\tilde{\psi}_1(\tilde{g}) = \tilde{\psi}_2(\tilde{g})$. Let U be an open n-disk containing $\{g_1, \dots, g_k\}$ such that $C_{f_1, f_2} \subset p(U)$ where $p \colon G \to M$ is the projection. Define $\varphi \colon U \to M$ by

$$\varphi(g) = (\tilde{\psi}_1(g))^{-1} \tilde{\psi}_2(g)\Gamma.$$

Then

$$\varphi(g) = e\Gamma \Leftrightarrow f_1(g\Gamma) = f_2(g\Gamma)$$

where e is the unit in G. Thus the set of roots of φ is given by $R_\varphi = \{g_1, \dots, g_k\}$.

Suppose that $g_i, g_j \in R_\varphi$, $i \neq j$ are Nielsen equivalent as roots of φ, i.e., there is a path α in U from g_i to g_j and a homotopy (rel. end points) $H \colon I \times I \to M$ such that $H_0 = \varphi \circ \alpha$ and $H_1 = e\Gamma$. Define

$$\widehat{H}(s, t) = \alpha(s)\tilde{\psi}_1(\alpha(s))H(s, t).$$

Then

$$\widehat{H}(s, 0) = \alpha(s)\tilde{\psi}_1(\alpha(s))\varphi(\alpha(s)) = f_2(\alpha(s)\Gamma)$$

and, similarly, $\widehat{H}(s, 1) = f_1(\alpha(s)\Gamma)$. Moreover,

$$\widehat{H}(0, t) = f_1(g_i\Gamma) = f_2(g_i\Gamma)$$

and

$$\widehat{H}(1, t) = f_1(g_j\Gamma) = f_2(g_j\Gamma).$$

This implies that $g_i\Gamma$ and $g_j\Gamma$ are Nielsen equivalent as coincidences of f_1 and f_2, contradicting the assumption that $\{g_1\Gamma, \dots, g_k\Gamma\}$ are distinct classes. Now $g_1, \dots, g_k$ are distinct root classes of φ and hence, by Theorem 2.3, $\omega(\varphi; g_i) = \omega(\varphi; g_j)$ for all $i, j = 1, \dots, k$.

Note that the map $F_i(g) = g\tilde{\psi}_i(g)$ is a lift of f_i for each $i = 1, 2$ and $F_1(\tilde{g}) = F_2(\tilde{g})$. Choose a euclidean neighborhood $\widetilde{U}$ of $\tilde{g}$ containing no other roots of φ and such that $p|\widetilde{U}$ is a homeomorphism. Consider the following commutative diagram

$$\begin{array}{ccccc}
\widetilde{U}, \widetilde{U}-\tilde{g} & \xrightarrow{(F_1, F_2)} & G\times G, G\times G-(p\times p)^{-1}(\Delta M) & \xleftarrow{l} & G, G-\Gamma \\
{\scriptstyle p|\widetilde{U}}\downarrow & & {\scriptstyle p\times p}\downarrow & & {\scriptstyle p}\downarrow \\
p(\widetilde{U}), p(\widetilde{U})-\tilde{g}\Gamma & \xrightarrow{(f_1, f_2)} & M\times M, M\times M-\Delta M & \xleftarrow{k} & M, M-e\Gamma
\end{array}$$

where $l(g) = (e, g)$ and $k(g\Gamma) = (e\Gamma, g\Gamma)$. Let $\sigma: G\times G,\ G\times G - (p\times p)^{-1}(\Delta M) \to G,\ G-\Gamma$ be given by $\sigma(g, g') = g^{-1}g'$.

Since $(F_1(g))^{-1}F_2(g) = (\tilde{\psi}_1(g))^{-1}\tilde{\psi}_2(g)$, it follows that

$$k\circ p\circ\sigma\circ(F_1, F_2) = k\circ\varphi.$$

Thus,

$$\begin{aligned}
\varphi^*\circ k^* &= (F_1, F_2)^*\circ\sigma^*\circ p^*\circ k^* \\
&= (F_1, F_2)^*\circ(p\times p)^* \quad (\text{since } l^*\cong, \sigma^* = (l^*)^{-1}) \\
&= (p|\widetilde{U})^*\circ(f_1, f_2)^*.
\end{aligned}$$

We have

$$\langle\varphi^*\circ k^*(\nu_M), o_{\tilde{g}}\rangle = \langle(p|\widetilde{U})^*\circ(f_1, f_2)^*(\nu_M), o_{\tilde{g}}\rangle,$$

which yields

$$\omega(\varphi; \tilde{g}) = \langle\varphi^*\mu_M, o_{\tilde{g}}\rangle = \langle(f_1, f_2)^*\nu_M, o_{\tilde{g}\Gamma}\rangle = I(f_1, f_2; \tilde{g}\Gamma)$$

where $\nu_M \in H^n(M\times M, M\times M-\Delta M)$ is the Thom class, $o_{\tilde{g}}$, $o_{\tilde{g}\Gamma}$ are the fundamental homology classes around $\tilde{g}$ and $\tilde{g}\Gamma$, respectively, and $I(f_1, f_2; \tilde{g}\Gamma)$ is the coincidence index of f_1 and f_2 at $\tilde{g}\Gamma$ [13, 15]. □

Lemma 3.2. *Let M be a compact connected nilmanifold. Suppose that $f, g: M \to M$ are maps such that $L(f, g)\neq 0$. Then each (essential) coincidence class is of index $+1$ or -1.*

Proof. Following [8], there exist an orientable principal T-bundle $T\to M\to N$, where T is a torus and N is a compact connected nilmanifold of dimension $\dim N < \dim M$, and fiber preserving maps f', g' such that

$$\begin{array}{ccc}
T & \xrightarrow{f'|T,\, g'|T} & T \\
\downarrow & & \downarrow \\
M & \xrightarrow{f',\, g'} & M \\
{\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\
N & \xrightarrow{\overline{f}',\, \overline{g}'} & N
\end{array}$$

is commutative and $f\sim f'$, $g\sim g'$.

By a general position argument, $\overline{f}'\sim\overline{\tau}$ and $\overline{g}'\sim\overline{\eta}$ such that the coincidence set $C_{\overline{\tau},\overline{\eta}}$ is finite. Then we can find maps $\tau, \eta: M\to M$ covering $\overline{\tau}, \overline{\eta}$ and $\tau\sim f'$, $\eta\sim g'$.

We may now assume without loss of generality that f and g are fiber preserving and $C_{\overline{f},\overline{g}}$ is finite. Let $b\in C_{\overline{f},\overline{g}}$, $T_b = p^{-1}(b)$, and $C_b\subset T_b$ be a coincidence class of $f|T_b$ and $g|T_b$. By [13, Theorem 3.3],

$$L(f, g) = L(f|T_b, g|T_b)\cdot L(\overline{f}, \overline{g}).$$

Since $L(f, g) \neq 0$, both $L(f|T_b, g|T_b)$ and $L(\overline{f}, \overline{g})$ are nonzero. Since the fiber T_b is a torus, it follows from [3, p. 125] that

$$\text{(IV)} \qquad I(f|T_b, g|T_b; C_b) = \pm 1.$$

Theorem 3.3 of [13] also gives the following local product formula:

$$\text{(V)} \qquad I(f, g; C_b) = I(f|T_b, g|T_b; C_b) \cdot I(\overline{f}, \overline{g}; b).$$

Let α_b be the coincidence class of $\overline{f}$ and $\overline{g}$ containing b and $\tilde{\alpha}$ be a coincidence class of f and g such that $p(\tilde{\alpha}) = \alpha_b$. If $b' \in \alpha_b$, then $I(f|T_{b'}, g|T_{b'}; C_{b'}) = I(f|T_b, g|T_b; C_b)$ by 3.1 (see also [3, p. 125]). By summing the indices of all $b' \in \alpha_b$ together with (IV) and (V), we obtain

$$\text{(VI)} \qquad I(f, g; \tilde{\alpha}) = \pm I(\overline{f}, \overline{g}; \alpha_b).$$

To complete the proof, we use induction on $m = \dim M$. For $m \leq 2$, the assertion follows from [3, p. 125]. Suppose that the assertion holds for dimension less than m. Since $\dim N < \dim M = m$, inductive hypothesis implies that $I(\overline{f}, \overline{g}; \alpha_b) = \pm 1$ and hence, by (VI), $I(f, g; \tilde{\alpha}) = \pm 1$. □

The following is a generalization of [1].

Theorem 3.3. *Let M be a compact connected nilmanifold. For any two maps $f, g: M \to M$,*

$$N(f, g) = |L(f, g)|.$$

Proof. This theorem is due to R. Brooks [3] for $\dim M \leq 2$. For $\dim M \geq 3$, it follows from Lemmas 3.1 and 3.2 since $N(f, g)$ is just the number of essential coincidence classes each of which has the same index that is either $+1$ or -1. □

Corollary 3.4. *Let (E, p, B) be an orientable fibration where $F = p^{-1}(b)$ $(b \in B)$ and E and B are compact connected nilmanifolds. For any given fiber preserving maps $f, g: E \to E$,*

$$N(f, g) = N(f_b, g_b) \cdot N(\overline{f}, \overline{g}).$$

Proof. This follows from the product formula (3.3) of [13] and Theorem 3.3 □

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