PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 2, June 1992

SAMPLE PATH-VALUED CONDITIONAL YEH-WIENER INTEGRALS AND A WIENER INTEGRAL EQUATION

CHULL PARK AND DAVID SKOUG

(Communicated by Lawrence T. Gray)

ABSTRACT. In this paper we evaluate the conditional Yeh-Wiener integral $E(F(x)|x(s, t) = \xi)$ for functions F of the form

$$F(x) = \exp\{\int_0^t \int_0^s \phi(\sigma, \tau, x(\sigma, \tau))\, d\sigma\, d\tau\}.$$

The method we use to evaluate this conditional integral is to first define a sample path-valued conditional Yeh-Wiener integral of the type $E(F(x)|x(s, \cdot) = \psi(\cdot))$ and show that it satisfies a Wiener integral equation. We next obtain a series solution for $E(F(x)|x(s, \cdot) = \psi(\cdot))$ by solving this Wiener integral equation. Finally, we integrate this series solution appropriately in order to evaluate $E(F(x)|x(s, t) = \xi)$.

1. INTRODUCTION

For $Q = [0, S] \times [0, T]$ let $C(Q)$ denote Yeh-Wiener space, i.e., the space of all real-valued continuous functions $x(s, t)$ on Q such that $x(0, t) = x(s, 0) = 0$ for every (s, t) in Q. Yeh [8] defined a Gaussian measure m_y on $C(Q)$ (later modified in [9]) such that as a stochastic process $\{x(s, t), (s, t) \in Q\}$ has mean $E[x(s, t)] = \int_{C(Q)} x(s, t) m_y(dx) = 0$ and covariance $E(x(s, t)x(u, v)) = \min\{s, u\}\min\{t, v\}$. Let $C_w \equiv C[0, T]$ denote the standard Wiener space on $[0, T]$ with Wiener measure m_w. In [10] Yeh introduced the concept of the conditional Wiener integral of F given X, $E(F|X)$, and for the case $X(x) = x(T)$ obtained some very useful results including a Kac-Feynman integral equation.

A very important class of functions in quantum mechanics are functions on $C[0, T]$ of the type

$$G(x) = \exp\left\{\int_0^T \theta(s, x(s))\, ds\right\}$$

where $\theta\colon [0, T] \times \mathbb{R} \to \mathbb{C}$.

Received by the editors October 22, 1990.
1991 *Mathematics Subject Classification*. Primary 60J65, 28C20.
Key words and phrases. Yeh-Wiener integral, conditional Yeh-Wiener integral, Wiener integral equation.
This paper was presented at the 97th annual meeting of the AMS at San Francisco on January 18, 1991.

Yeh's result [10] shows that under suitable regularity conditions on θ, the conditional Wiener integral

$$H(t,\xi)=(2\pi t)^{-1/2}\exp\left\{-\frac{\xi^2}{2t}\right\}E\left(\exp\left\{\int_0^t \theta(s,x(s))\,ds\right\}|x(t)=\xi\right) \tag{1.1}$$

satisfies the Kac-Feynman integral equation

$$\begin{aligned}H(t,\xi)&=(2\pi t)^{-1/2}\exp\left\{-\frac{\xi^2}{2t}\right\}\\&\quad+\int_0^t[2\pi(t-s)]^{-1/2}\int_{\mathbb{R}}\theta(s,\eta)H(s,\eta)\exp\left\{-\frac{(\eta-\xi)^2}{2(t-s)}\right\}d\eta\,ds\end{aligned} \tag{1.2}$$

whose solution can be expressed as an infinite series of terms involving Lebesgue integrals. Then using (1.1), one can use the series solution of (1.2) to evaluate the conditional Wiener integral

$$E\left(\exp\left\{\int_0^t \theta(s,x(s))\,ds\right\}|x(t)=\xi\right).$$

The main purpose of this paper is to evaluate the conditional Yeh-Wiener integral

$$E\left(\exp\left\{\int_0^t\int_0^s \phi(\sigma,\tau,x(\sigma,\tau))\,d\sigma\,d\tau\right\}|x(s,t)=\xi\right) \tag{1.3}$$

in terms of Lebesgue integrals.

One possible approach to this problem would be to find an integral equation involving the expression (1.3) and then to solve this integral equation. Various attempts trying this approach have not been very successful because of the difficulty involved in solving the resulting integral equation; as an example see [2, Theorem 2.1] where Chang, Ahn, and Chang derive an integral equation involving the expression (1.3).

The approach we use in this paper to evaluate (1.3) is to first define a sample path-valued conditional Yeh-Wiener integral of the type

$$E\left(\exp\left\{\int_0^t\int_0^s \phi(\sigma,\tau,x(\sigma,\tau))\,d\sigma\,d\tau\right\}|x(s,\cdot)=\psi(\cdot)\right), \tag{1.4}$$

which satisfies a Wiener integral equation similar to that of Cameron and Storvick [1]. The Wiener integral equation is then solved to evaluate (1.4), and finally by integrating (1.4) appropriately, we evaluate (1.3).

2. Sample path-valued conditional expectation

Throughout this paper we assume that $\psi(t)$ is a real-valued continuous function on $[0,T]$ vanishing at the origin; that is to say, ψ is in $C_w\equiv C[0,T]$.

For a Yeh-Wiener integrable function $F(x)$, consider the conditional Yeh-Wiener integral of the type

$$E(F(x)|x(S,\cdot)=\psi(\cdot)). \tag{2.1}$$

Since $x(s,\cdot)-(s/S)x(S,\cdot)$ and $x(S,\cdot)$ are stochastically independent processes, we have as usual,

$$E(F(x)|x(S,\cdot)=\psi(\cdot))=E\left[F(x(*,\cdot)-\frac{*}{S}x(S,\cdot)+\frac{*}{S}\psi(\cdot))\right] \tag{2.2}$$

for almost all $\psi\in C_w$.

Let $y(\cdot)$ be a tied-down Brownian motion, namely,

$$\{y(t),\ 0 \le t \le T\} = \{w \in C_w : w(T) = \xi\}.$$

Then, as is well known, $y(\cdot)$ can be expressed in terms of the standard Wiener process,

$$y(t) = w(t) - \frac{t}{T}w(T) + \frac{t}{T}\xi.$$

Next we establish the following important theorem that will play a key role in this paper.

Theorem 1. *If* $F \in L^1(C(Q), m_y)$, *then*

(i) $E_w\{E(F(x)|x(S, \cdot) = \sqrt{S}w(\cdot))\} = E[F(x)]$, *and*

(ii) $E_w\{E(F(x)|x(S, \cdot) = \sqrt{S}(w(\cdot) - \frac{\cdot}{T}w(T) + \frac{\cdot}{T}\xi))\} = E(F(x)|x(S, T) = \sqrt{S}\xi)$.

Proof. (i) Using (2.2), we may write

$$\begin{aligned} &E_w\{E_x(F(x)|x(S, \cdot) = \sqrt{S}w(\cdot))\} \\ &\qquad = E_w\left\{E\left[F\left(x(*, \cdot) - \frac{*}{S}x(S, \cdot) + \frac{*}{\sqrt{S}}w(\cdot)\right)\right]\right\}. \end{aligned}$$

Let $y(s, t) = x(s, t) - (s/S)x(S, t) + (s/\sqrt{S})w(t)$. Then $E[y(s, t)] = 0$ and the covariance of y, $E[y(s, t)y(u, v)] = \min\{s, u\}\min\{t, v\}$. Hence, $y(s, t)$ is the Yeh-Wiener process, and so

$$E_w\left\{E\left[F\left(x(*, \cdot) - \frac{*}{S}x(S, \cdot) + \frac{*}{\sqrt{S}}w(\cdot)\right)\right]\right\} = \int_{C(Q)} F(y)m_y(dy) = E[F(x)].$$

(ii) Using Theorem 12 in [7], we may write

$$E(F(x)|x(S, T) = \sqrt{S}\xi) = E\left[F\left(x(*, \cdot) - \frac{(*)}{S}\frac{(\cdot)}{T}x(S, T) + \frac{(*)(\cdot)}{\sqrt{S}T}\xi\right)\right]. \tag{2.3}$$

The right-hand side of (2.3) can be written in the form

$$E\left[F\left(x(*, \cdot) - \frac{*}{S}x(S, \cdot) + \frac{*}{S}\left(x(S, \cdot) - \frac{\cdot}{T}x(S, T) + \frac{\cdot}{T}\sqrt{S}\xi\right)\right)\right]. \tag{2.4}$$

Note that $x(s, t) - (s/S)x(S, t)$ and $x(S, t) - (t/T)x(S, T)$ are independent processes and $x(S, \cdot) - (\cdot/T)x(S, T)$ is equivalent to the process $\sqrt{S}[W(\cdot) - (\cdot/T)w(T)]$ where $w(\cdot)$ is the standard Wiener process. Therefore the expression in (2.4) equals

$$\begin{aligned} &E\left[F\left(x(*, \cdot) - \frac{*}{S}x(S, \cdot) + \frac{*}{\sqrt{S}}\left(w(\cdot) - \frac{\cdot}{T}w(T) + \frac{\cdot}{T}\xi\right)\right)\right] \\ &\quad = E_w\left\{E_x\left[F\left(x(*, \cdot) - \frac{*}{S}x(S, \cdot) + \frac{*}{\sqrt{S}}\left(w(\cdot) - \frac{\cdot}{T}w(T) + \frac{\cdot}{T}\xi\right)\right)\right]\right\} \\ &\quad = E_w\left\{E\left(F(x)|x(S, \cdot) = \sqrt{S}\left(w(\cdot) - \frac{\cdot}{T}w(T) + \frac{\cdot}{T}\xi\right)\right)\right\}, \end{aligned}$$

which was to be shown.

3. Examples

Example 1. For $x \in C(Q)$, let $F(x) = \int_Q x(s, t)\,ds\,dt$. Then by (2.2) and the Fubini Theorem,

$$\begin{aligned} E&\left(\int_Q x(s, t)\,ds\,dt | x(S, \cdot) = \psi(\cdot)\right) \\ &= E\left[\int_Q \left\{x(s, t) - \frac{s}{S}x(S, t) + \frac{s}{S}\psi(t)\right\} ds\,dt\right] \\ &= \int_Q \frac{s}{S}\psi(t)\,ds\,dt = \frac{S}{2}\int_0^T \psi(t)\,dt. \end{aligned}$$

Example 2. Let $F(x) = \int_Q x^2(s, t)\,ds\,dt$. Then by (2.2) and the Fubini Theorem,

$$\begin{aligned} I \equiv E&\left(\int_Q x^2(s, t)\,ds\,dt | x(S, \cdot) = \psi(\cdot)\right) \\ &= E\left[\int_Q \left\{x(s, t) - \frac{s}{S}x(S, t) + \frac{s}{S}\psi(t)\right\}^2 ds\,dt\right] \\ &= \int_Q E_x\left[\left(x(s, t) - \frac{s}{S}x(S, t)\right)^2 + \frac{s^2}{S^2}\psi^2(t)\right. \\ &\qquad \left. + \frac{2s}{S}\psi(t)\left(x(s, t) - \frac{s}{S}x(S, t)\right)\right] ds\,dt. \end{aligned}$$

Using the fact that $E[x(u, v)] = 0$ for every $(u, v) \in Q$ and that

$$E[x(s, t)x(u, v)] = \min\{s, u\}\min\{t, v\},$$

we obtain

$$\begin{aligned} I &= \int_Q \left(st - \frac{s^2 t}{S} + \frac{s^2}{S^2}\psi^2(t)\right) ds\,dt \\ &= \frac{S^2T^2}{4} - \frac{S^2T^2}{6} + \frac{S}{3}\int_0^T \psi^2(t)\,dt = \frac{S^2T^2}{12} + \frac{S}{3}\int_0^T \psi^2(t)\,dt. \end{aligned}$$

Furthermore, if we replace $\psi(\cdot)$ by $\sqrt{S}w(\cdot)$ and integrate in w over C_w, we obtain that

$$E_w(I) = \frac{S^2T^2}{12} + \frac{S^2}{3}E\left[\int_0^T w^2(t)\,dt\right] = \frac{S^2T^2}{12} + \frac{S^2}{3}\frac{T^2}{2} = \frac{S^2T^2}{4},$$

which agrees with the quantity $E[\int_Q x^2(s, t)\,ds\,dt] = E[F(x)]$.

Example 3. For $x \in C(Q)$ let $F(x) = \exp\{\int_Q x(s, t)\,ds\,dt\}$. Then, due to the fact that $\int_Q x(s,t)\,ds\,dt = \int_Q (T-t)(S-s)\,dx(s,t)$, we obtain

$$\begin{aligned} J &\equiv E\left(\exp\left\{\int_Q x(s,t)\,ds\,dt\right\} |x(S,\cdot) = \psi(\cdot)\right) \\ &= E\left[\exp\left\{\int_Q (T-t)(S-s)\,d\left(x(s,t) - \frac{s}{S}x(S,t) + \frac{s}{S}\psi(t)\right)\right\}\right] \\ &= \exp\left\{\int_0^S \frac{S-s}{S}\,ds \int_0^T (T-t)\,d\psi(t)\right\} \\ &\quad \times E\left[\exp\left\{\int_Q (T-t)\left(\frac{S}{2}-s\right)\,dx(s,t)\right\}\right], \end{aligned}$$

where we also used the fact that

$$\int_Q (T-t)(S-s)\,d(sx(S,t)) = \frac{S^2}{2}\int_0^T (T-t)\,dx(S,t) = \frac{S^2}{2}\int_Q (T-t)\,dx(s,t).$$

Thus

(3.1)
$$J = \exp\left\{\frac{S}{2}\int_0^T (T-t)\,d\psi(t)\right\}\exp\left\{\frac{S^3T^3}{72}\right\} = \exp\left\{\frac{S}{2}\int_0^T \psi(t)\,dt + \frac{S^3T^3}{72}\right\}.$$

Furthermore, if we replace $\psi(\cdot)$ by $\sqrt{S}w(\cdot)$ and integrate in w over C_w, we obtain that

$$\begin{aligned} E_w(J) &= \exp\left\{\frac{S^3T^3}{72}\right\}\int_{C_w} \exp\left\{\frac{S}{2}\int_0^T (T-t)\,d(\sqrt{S}w(t))\right\} m_w(dw) \\ &= \exp\left\{\frac{S^3T^3}{72} + \frac{S^3T^3}{24}\right\} = \exp\left\{\frac{S^3T^3}{18}\right\}, \end{aligned}$$

which agrees with the quantity $E[\exp\{\int_Q x(s,t)\,ds\,dt\}] = E[F(x)]$.

On the other hand, if we replace $\psi(\cdot)$ by $\sqrt{S}(w(\cdot) - (\cdot/T)w(T) + (\cdot/T)\xi)$ and integrate in w over C_w, using (3.1) we obtain that

$$\begin{aligned} &E_w\left\{E\left(\exp\left[\int_Q x(s,t)\,ds\,dt\right]\middle| x(S,\cdot) = \sqrt{S}\left(w(\cdot) - \frac{\cdot}{T}w(T) + \frac{\cdot}{T}\xi\right)\right)\right\} \\ &= \exp\left\{\frac{S^3T^3}{72}\right\} E_w\left[\exp\left\{\frac{S^{3/2}}{2}\int_0^T (T-t)\,d\left(w(t) - \frac{t}{T}w(T) + \frac{t}{T}\xi\right)\right\}\right] \\ &= \exp\left\{\frac{S^3T^3}{72} + \frac{S^{3/2}T\xi}{4}\right\} E\left[\exp\left\{\frac{S^{3/2}}{2}\int_0^T \left(\frac{T}{2}-t\right)\,dw(t)\right\}\right] \\ &= \exp\left\{\frac{S^3T^3}{72} + \frac{S^{3/2}T\xi}{4} + \frac{S^3T^3}{96}\right\} = \exp\left\{\frac{7S^3T^3}{288} + \frac{S^{3/2}T\xi}{4}\right\}. \end{aligned}$$

Also,

$$
\begin{aligned}
E&\left(\exp\left\{\int_Q x(s,t)\,ds\,dt\right\}\middle| x(S,T)=\sqrt{S}\xi\right)\\
&=E\left(\exp\left\{\int_Q (T-t)(S-s)\,dx(s,t)\right\}\middle| x(S,T)=\sqrt{S}\xi\right)\\
&=E\left[\exp\left\{\int_Q (T-t)(S-s)\,d\left(x(s,t)-\frac{st}{ST}x(S,T)+\frac{st}{ST}\sqrt{S}\xi\right)\right\}\right]\\
&=\exp\left\{\frac{S^{3/2}T\xi}{4}\right\}E\left[\exp\left\{\int_Q\left[(T-t)(S-s)-\frac{ST}{4}\right]dx(s,t)\right\}\right]\\
&=\exp\left\{\frac{S^{3/2}T\xi}{4}+\frac{7S^3T^3}{288}\right\}.
\end{aligned}
$$

Thus, we have verified directly that (ii) of Theorem 1 holds for the function $F(x)=\exp\{\int_Q x(s,t)\,ds\,dt\}$.

4. Evaluation of $E(\exp\{\int_0^S\int_0^T \phi(\sigma,\tau,x(\sigma,\tau))\,d\tau\,d\sigma\}|x(S,T)=\sqrt{S}\xi)$

Let $\phi(s,t,u)$ be a bounded continuous function on $Q\times\mathbb{R}$, and let

$$\theta(\sigma,x(\sigma,\cdot))=\int_0^T \phi(\sigma,\tau,x(\sigma,\tau))\,d\tau.$$

Then

$$
\begin{aligned}
F(s,x)&\equiv\exp\left\{\int_0^s\int_0^T \phi(\sigma,\tau,x(\sigma,\tau))\,d\tau\,d\sigma\right\}\\
&=\exp\left\{\int_0^s \theta(\sigma,x(\sigma,\cdot))\,d\sigma\right\}.
\end{aligned}
$$

Since $\partial F(s,x)/\partial s=\theta(s,x(s,\cdot))F(s,x)$, by integrating over $[0,s]$, $0<s\le S$, we obtain

$$F(s,x)-1=\int_0^s \theta(\sigma,x(\sigma,\cdot))F(\sigma,x)\,d\sigma.$$

Next we take a conditional expectation of both sides and then use the Fubini Theorem to obtain

(4.1)

$$E(F(s,x)|x(s,\cdot)=\psi(\cdot))=1+\int_0^s E(\theta(\sigma,x(\sigma,\cdot))F(\sigma,x)|x(s,\cdot)=\psi(\cdot))\,d\sigma.$$

Now for $0<\sigma\le s\le S$,

$$
\begin{aligned}
(4.2)\qquad & E(\theta(\sigma,x(\sigma,\cdot))F(\sigma,x)|x(s,\cdot)=\psi(\cdot))\\
&=E\Big[\theta(\sigma,x(\sigma,\cdot)-\frac{\sigma}{s}x(s,\cdot)+\frac{\sigma}{s}\psi(\cdot))\\
&\qquad\times\exp\Big\{\int_0^\sigma \theta\Big(u,x(u,\cdot)-\frac{u}{\sigma}x(\sigma,\cdot)\\
&\qquad\qquad+\frac{u}{\sigma}\Big[x(\sigma,\cdot)-\frac{\sigma}{s}x(s,\cdot)+\frac{\sigma}{s}\psi(\cdot)\Big]\Big)\,du\Big\}\Big].
\end{aligned}
$$

Note that for $0 \le u \le \sigma$, $x(u, \cdot) - (u/\sigma)x(\sigma, \cdot)$ and $x(\sigma, \cdot) - (\sigma/s)x(s, \cdot)$ are independent processes, and $x(\sigma, \cdot) - (\sigma/s)x(s, \cdot)$ is equivalent to $\sqrt{\sigma(1-(\sigma/s))}w(\cdot)$ for fixed σ and s. Therefore, it follows from (4.2) that

$$
\begin{aligned}
(4.3)\ & E(\theta(\sigma, x(\sigma, \cdot))F(\sigma, x)|x(s, \cdot) = \psi(\cdot)) \\
&= E_w\left\{\theta\left(\sigma, \sqrt{\sigma\left(1-\frac{\sigma}{s}\right)}w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right)\right. \\
&\qquad \times E_x\left[\exp\left\{\int_0^\sigma \theta\left(u, x(u, \cdot)\right.\right.\right. \\
&\qquad\qquad \left.\left.\left.\left. - \frac{u}{\sigma}x(\sigma, \cdot) + \frac{u}{\sigma}\left[\sqrt{\sigma\left(1-\frac{\sigma}{s}\right)}w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right]\right)\, du\right\}\right]\right\} \\
&= E_w\left[\theta\left(\sigma, \sqrt{\sigma\left(1-\frac{\sigma}{s}\right)}w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right)\right. \\
&\qquad \times E_x\left(\exp\left\{\int_0^\sigma \theta(u, x(u, \cdot))\, du\right\}\middle| x(\sigma, \cdot)\right. \\
&\qquad\qquad \left.\left. = \sqrt{\sigma\left(1-\frac{\sigma}{s}\right)}w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right)\right] \\
&= E_w\left[\theta\left(\sigma, \sqrt{\sigma\left(1-\frac{\sigma}{s}\right)}w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right)\right. \\
&\qquad \left. \times E_x\left(F(\sigma, x)|x(\sigma, \cdot) = \sqrt{\sigma\left(1-\frac{\sigma}{s}\right)}w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right)\right].
\end{aligned}
$$

If we set

$$
(4.4) \qquad G(s, \psi(\cdot)) \equiv E(F(s, x)|x(s, \cdot) = \psi(\cdot)),
$$

then substituting (4.3) into equation (4.1), we find that $G(s, \psi(\cdot))$ satisfies the Wiener integral equation

$$
\begin{aligned}
(4.5)\quad G(s, \psi(\cdot)) = 1 + \int_0^s E_w&\left[\theta\left(\sigma, \sqrt{\sigma\left(1-\frac{\sigma}{s}\right)}w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right)\right. \\
&\left. \times G\left(\sigma, \sqrt{\sigma\left(1-\frac{\sigma}{s}\right)}w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right)\right] d\sigma.
\end{aligned}
$$

This integral equation is very similar to the Cameron-Storvick integral equation [1, equation (4.3)]. But most importantly it is easy to see that the series solution to (4.5) is given by

$$
(4.6) \qquad G(s, \psi(\cdot)) = \sum_{k=0}^{\infty} H_k(s, \psi(\cdot)),
$$

where the sequence $\{H_k\}$ is given inductively by $H_0(s, \psi(\cdot)) = 1$ and

$$
\begin{aligned}
H_{k+1}(s, \psi(\cdot)) = \int_0^s E_w&\left\{\theta\left(\sigma, \sqrt{\sigma\left(1-\frac{\sigma}{s}\right)}w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right)\right. \\
&\left. \times H_k\left(\sigma, \sqrt{\sigma\left(1-\frac{\sigma}{s}\right)}w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right)\right\} d\sigma.
\end{aligned}
$$

Furthermore, if $|\theta(s, \xi)| \le M$ on $[0, S] \times \mathbb{R}$ then one can easily verify by induction that

$$|H_k(s, \psi(\cdot))| \le \frac{(Ms)^k}{k!} \le \frac{(MS)^k}{k!} \quad \text{for } k = 0, 1, 2, \dots,$$

hence $\sum_{k=0}^{\infty} |H_k(s, \psi(\cdot))| \le \exp\{MS\}$, and so the series $\sum_{k=0}^{\infty} H_k(s, \psi(\cdot))$ converges uniformly on $[0, S]$. Thus, it is immediate that $G(s, \psi(\cdot))$ given by (4.6) is a bounded continuous solution of (4.5). To show that it is the only bounded continuous solution of (4.5), assume that G_1 and G_2 are each such solutions of (4.5). Then $H = G_2 - G_1$ is a bounded continuous solution of

$$\text{(4.7)} \qquad \begin{aligned} H(s, \psi(\cdot)) = \int_0^s E_w \Big\{ &\theta\left(\sigma, \sqrt{\sigma\left(1 - \tfrac{\sigma}{s}\right)} w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right) \\ &\times H\left(\sigma, \sqrt{\sigma\left(1 - \tfrac{\sigma}{s}\right)} w(\cdot) + \frac{\sigma}{s}\psi(\cdot)\right)\Big\}\, d\sigma \\ \equiv L\{H(s, \psi(\cdot))\}&. \end{aligned}$$

Note that equation (4.7) implies that

$$\text{(4.8)} \qquad H(s, \psi(\cdot)) = L^n\{H(s, \psi(\cdot))\}$$

for $n = 1, 2, \dots$. Using induction on n as before with $|H(s, \xi)| \le N$ on $[0, S] \times \mathbb{R}$, it is immediate that

$$|H(s, \psi(\cdot))| = |L^n\{H(s, \psi(\cdot))\}| \le (NS)^n/n!$$

for $n = 1, 2, \dots$. Thus, we may conclude that $H(s, \psi(\cdot)) \equiv 0$, which establishes the uniqueness of a bounded continuous solution of (4.5).

It is now easy to evaluate

$$\text{(4.9)} \qquad I \equiv E\left(\exp\left\{\int_0^S \int_0^T \phi(\sigma, \tau, x(\sigma, \tau))\, d\tau\, d\sigma\right\}\middle| x(S, T) = \sqrt{S}\xi\right)$$

under the assumption that ϕ is bounded and continuous. By Theorem 1, (4.4), and (4.6) we have that

$$\text{(4.10)} \qquad \begin{aligned} I &= E_w\left\{E\left(F(S, x) | x(S, \cdot) = \sqrt{S}\left(w(\cdot) - \frac{\cdot}{T}w(T) + \frac{\cdot}{T}\xi\right)\right)\right\} \\ &= E_w\left[G\left(S, \sqrt{S}\left(w(\cdot) - \frac{\cdot}{T}w(T) + \frac{\cdot}{T}\xi\right)\right)\right] \\ &= \sum_{k=0}^{\infty} E_w\left[H_k\left(S, \sqrt{S}\left(w(\cdot) - \frac{\cdot}{T}w(T) + \frac{\cdot}{T}\xi\right)\right)\right]. \end{aligned}$$

But now each Wiener integral in the summand can be expressed in terms of Lebesgue integrals in the usual way as Cameron and Storvick demonstrated in [1].

References

1. R. H. Cameron and D. A. Storvick, *An operator valued Yeh-Wiener integral and a Wiener integral equation*, Indiana Univ. Math. J. **25** (1976), 235–258.
2. K. S. Chang, J. M. Ahn, and J. S. Chang, *An evaluation of the conditional Yeh-Wiener integral*, Pacific J. Math. **124** (1986), 107–117.

3. D. M. Chung and J. M. Ahn, *Conditional Yeh-Wiener integrals*, J. Korean Math. Soc., **20** (1983), 209–221.
4. J. L. Doob, *Stochastic processes*, Wiley, New York, 1965.
5. C. Park, *On Fredholm transformations in Yeh-Wiener space*, Pacific J. Math. **40** (1972), 173–195.
6. C. Park and D. L. Skoug, *A simple formula for conditional Wiener integrals with applications*, Pacific J. Math. **135** (1988), 381–394.
7. ——, *Conditional Yeh-Wiener integrals with vector-valued conditioning functions*, Proc. Amer. Math. Soc. **105** (1989), 450–461.
8. J. Yeh, *Wiener measure in a space of functions of two variables*, Trans. Amer. Math. Soc., **95** (1960), 433–450.
9. ——, *Stochastic processes and the Wiener integral*, Marcel Dekker, New York, 1983.
10. ——, *Inversion of conditional Wiener integrals*, Pacific J. Math. **59** (1975), 623–638.

Department of Mathematics & Statistics, Miami University, Oxford, Ohio 45056

Department of Mathematics & Statistics, University of Nebraska, Lincoln, Nebraska 68588-0323

E-mail address: cpark@miavx1.acs.muohio.edu