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RATIONAL MODULES AND CAUCHY TRANSFORMS, II

JAMES LI-MING WANG

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ABSTRACT. We apply the higher order Cauchy transform to describe the closures of rational modules with respect to the uniform norm, the L^p norm, and the BMO norm on a compact set in the plane.

Let X be a compact subset of the complex plane $\mathbb{C}$. We denote by $\mathscr{R}(X)$ the space of all rational functions with poles off X and by $\mathscr{R}(X)\overline{\mathscr{P}}_1$ the rational module $\{r_0(z)+\bar{z}r_1(z)\}$, where each $r_i(z) \in \mathscr{R}(X)$. Let $\bar{\partial} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ be the usual Cauchy-Riemann operator in the complex plane and let $\bar{\partial}^2 = \bar{\partial} \circ \bar{\partial}$. We note that a function satisfies $\bar{\partial}^2 f = 0$ in an open set if and only if $f = h + \bar{z}k$ with h and k analytic. A major problem in approximation theory is to describe the closure of the rational module $\mathscr{R}(X)\overline{\mathscr{P}}_1$ in various norms (see [4]). It has been shown in [6] that $\mathscr{R}(X)\overline{\mathscr{P}}_1$ is dense in $C(X)$, the space of all continuous functions on X, for any compact set X when $\dot{X}$, the interior of X, is empty. However, the presence of an interior really complicates the situation. An outstanding open problem is whether $\mathscr{R}(X)\overline{\mathscr{P}}_1$ is dense in the space $\{f \in C(X): \bar{\partial}^2 f = 0 \text{ in } \dot{X}\}$ for all compact sets X.

For $1 \le p < \infty$, let $L^p(X) = L^p(X, dm)$ with dm the two-dimensional Lebesgue measure on $\mathbb{C}$ and let $L^p_a(X)$ be the closed subspace of $L^p(X)$ that consists of functions analytic in $\dot{X}$. If V is any space of functions on X, we denote by $[V]_p$ and $[V]_u$ the closure of V with respect to the $L^p(X)$ norm and the uniform norm on X, respectively, and we write $\widehat{V} = \{\hat{f}: f \in V\}$ where $\hat{f}$ is the usual Cauchy transform of f,

$$\hat{f}(z) = \int \frac{f(\zeta)}{\zeta - z}\, dm(\zeta).$$

In [7] the author proved that $[\mathscr{R}(X)\overline{\mathscr{P}}_1]_p = [\mathscr{R}(X) + L^p_a(X)^\wedge]_p$ for all $1 \le p < \infty$, among other things. In this note, we show that more functions of the type of Cauchy transform belong to the closure of the rational module with respect to the uniform norm and the L^p norms, and then extend the result to the BMO norm case.

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To each compact subset X of $\mathbb{C}$ $(\cong \mathbb{R}^2)$ one associates the restrictive spaces

$$\mathrm{BMO}(X) = \mathrm{BMO}(\mathbb{C})|_X \cong \mathrm{BMO}(\mathbb{C})/I(X),$$
$$\mathrm{VMO}(X) = \mathrm{VMO}(\mathbb{C})|_X \cong \mathrm{VMO}(\mathbb{C})/J(X),$$

where

$$I(X) = \{f \in \mathrm{BMO}(\mathbb{C}) \colon f = 0 \text{ a.e. on } X\},$$
$$J(X) = \{f \in \mathrm{VMO}(\mathbb{C}) \colon f = 0 \text{ a.e. on } X\}.$$

Let $[\mathscr{R}(X)\overline{\mathscr{P}}_1]_{\mathrm{BMO}}$ and $[\mathscr{R}(X) + L_a^2(X)^\frown]_{\mathrm{BMO}}$ be the closure in $\mathrm{BMO}(X)$ of $\mathscr{R}(X)\overline{\mathscr{P}}_1$ and $\mathscr{R}(X) + L_a^2(X)^\frown$, respectively. Holden's extension theorem [2] implies that $[\mathscr{R}(X)\overline{\mathscr{P}}_1]_{\mathrm{BMO}}$ and $[\mathscr{R}(X) + L_a^2(X)^\frown]_{\mathrm{BMO}}$ are contained in $\mathrm{VMO}(X)$, and consequently in

$$\mathrm{VMO}(X) \cap \{f \colon \overline{\partial}^2 f = 0 \text{ in } \dot{X}\}.$$

We can now state our main result.

Theorem 1. *Let X be a compact set of the complex plane. Then*

(i) $[\mathscr{R}(X)\overline{\mathscr{P}}_1]_u = [\mathscr{R}(X) + L_a^p(X)^\frown]_u$ *for all* $2 < p < \infty$.

(ii) $[\mathscr{R}(X)\overline{\mathscr{P}}_1]_p = [\mathscr{R}(X) + L_a^{p'}(X)^\frown]_p$ *for all* $1 \le p < \infty$. *Here p' is a real number such that $1/p' = 1/p + 1/2$ if $2 < p < \infty$; p' is any real number > 1 if $p = 2$ and $p' = 1$ if $1 \le p < 2$.*

(iii) $[\mathscr{R}(X)\overline{\mathscr{P}}_1]_{\mathrm{BMO}} = [\mathscr{R}(X) + L_a^2(X)^\frown]_{\mathrm{BMO}}$.

It is well known that $[\mathscr{R}(X)\overline{\mathscr{P}}_1]_u = [\mathscr{R}(X) + \mathscr{R}(X)^\frown]_u$, $[\mathscr{R}(X)\overline{\mathscr{P}}_1]_p = \{\mathscr{R}(X) + \mathscr{R}(X)^\frown]_p$ for all $1 \le p < \infty$, and $[\mathscr{R}(X)\overline{\mathscr{P}}_1]_{\mathrm{BMO}} = [\mathscr{R}(X) + \mathscr{R}(X)^\frown]_{\mathrm{BMO}}$ (see [7]). Therefore, it suffices to prove the other inclusions.

Proof of Theorem 1. (i) Let μ be any complex Borel measure that annihilates $\mathscr{R}(X)\overline{\mathscr{P}}_1$. We shall show that μ annihilates $L_a^p(X)^\frown$ for any p, $2 < p < \infty$.

We write $\check{\mu}(z) = \int (\overline{\zeta} - \overline{z})/(\zeta - z)\, d\mu(\zeta)$. We note that $\check{\mu}$ is continuous; $\check{\mu} = 0$ off X and $\overline{\partial}\check{\mu} = \hat{\mu}$ in the sense of distribution [6]. It follows that $\check{\mu} = 0$ a.e. on ∂X, the boundary of X. Now $\hat{\mu} \in L^{1+\varepsilon}(X)$, $0 < \varepsilon < 1$, and so it follows from the theory of singular integrals [5] that $\check{\mu}$ is absolutely continuous on almost every line parallel to each of the coordinate axes and that $\partial\check{\mu}/\partial x$ and $\partial\check{\mu}/\partial y$ exist almost everywhere in the usual sense. By a lemma of Schwartz (see [1]), these derivatives coincide with the corresponding distribution derivatives and so $\hat{\mu} = \overline{\partial}\check{\mu}$ almost everywhere in the usual sense and $\hat{\mu} = 0$ almost everywhere on ∂X. Now we are ready to prove that $\int_{\dot{X}} \hat{f}\, d\mu = 0$ for all $f \in L_a^p(X)$, $2 < p < \infty$.

Again, a similar construction in [1] will give a sequence of function ρ_n, $n = 1, 2, \ldots$ such that ρ_n has compact support in $\dot{X}$ and $\overline{\partial}\rho_n \to \hat{\mu}$ in the $L^q(X)$ norm for all $1 \le q < 2$. Writing $\check{\mu}$ in the form $\check{\mu} = (k_1 - k_2) + i(k_3 - k_4)$, where each k_j is continuous, nonnegative, and zero on ∂X, we can set $\rho_n^j = -\pi^{-1}\sup(k_j - n^{-1}, 0)$. Then the desired sequence ρ_n, $n = 1, 2, \ldots$ is obtained by taking $\rho_n = (\rho_n^1 - \rho_n^2) + i(\rho_n^3 - \rho_n^4)$.

Thus for any f in $L_a^p(X)$, $2 < p < \infty$,

$$0 = \lim_{n\to\infty} \int_{\dot{X}} f(\overline{\partial}\rho_n)\, dm = \int_{\dot{X}} f\hat{\mu}\, dm = \int_X f\hat{\mu}\, dm = \int_X \hat{f}\, d\mu,$$

and this completes the proof of (i).

(ii) We will treat the case $2 \le p < \infty$ only, the other case will follow similarly. Let g be any function in $L^q(X)$, $1 < q \le 2$, $p^{-1} + q^{-1} = 1$ such that g annihilates $\mathscr{R}(X)\overline{\mathscr{P}}_1$. We shall show that g annihilates $L_a^{p'}(X)^\frown$.

We write $\check{g} = (g\,dm)^\smile$. Using a similar argument as in the proof of (i), we conclude that $\hat{g} = \overline{\partial}\check{g} = 0$, almost everywhere on ∂X. The Sobolev's theorem implies $\hat{g} \in L^{q'}(X)$ where $1/q' = 1/q - 1/2$ if $1 < q < 2$, and $q' < \infty$ is any real number if $q = 2$. Again, a similar construction gives a sequence of functions ρ_n, $n = 1, 2, \dots$ such that each ρ_n has support in $\dot{X}$ on $\overline{\partial}\rho_n \to \hat{g}$ in the $L^{q'}(X)$ norm. Thus for any f in $L_a^{p'}(X)$, $p'^{-1} + q'^{-1} = 1$, we have

$$0 = \lim_{n\to\infty} \int_{\dot{X}} f(\overline{\partial}\rho_n)\,dm = \int_{\dot{X}} f\hat{g}\,dm = \int_X f\hat{g}\,dm = \int_X \hat{f} g\,dm,$$

and this completes the proof of (ii).

(iii) Let $f \in [\mathscr{R}(X) + L_a^2(X)^\frown]_{\mathrm{BMO}}$. We can think that in fact $f \in \mathrm{VMO}(\mathbb{C})$ and $\overline{\partial}^2 f = 0$ in $\dot{X}$. Replacing f by ϕf where $\phi \in C_c^\infty(\mathbb{C})$ takes the value 1 on a neighborhood of X, we can assume also that f is compactly supported. It is also easy to see that [3] $\mathrm{VMO}(X) = \mathrm{CMO}(\mathbb{C})/K(X)$, where $\mathrm{CMO}(\mathbb{C})$ is the closure of $C_c^\infty(\mathbb{C})$ in $\mathrm{BMO}(\mathbb{C})$ and $K(X) = \{f \in \mathrm{CMO}(\mathbb{C}) \colon f = 0$ a.e. on $X\}$. Since $\mathrm{CMO}(\mathbb{C})^* = H^1(\mathbb{C})$, it is clear that $\mathrm{VMO}(X)^* = \{h \in H^1(\mathbb{C}) \colon h = 0$ a.e. on $X^c\}$.

Let $h \in \mathrm{VMO}(X)^*$ such that h annihilates $\mathscr{R}(X)\overline{\mathscr{P}}_1$. We must show that h also annihilates $L_a^2(X)^\frown$. We note that $\hat{h} = \overline{\partial}\check{h} = 0$, almost everywhere on ∂X, and $\hat{h}$ belongs to the Sobolev space $L_1^1(X)$ and thus belongs to $L^2(X)$ because of the theory of singular integrals and the Sobolev's theorem (see [5]). Let $f \in L_a^2(X)$. The same construction and similar proof as in part (i) gives us a sequence of functions ρ_n, $n = 1, 2, \dots$ such that each ρ_n has support in $\dot{X}$ and $\overline{\partial}\rho_n \to \hat{h}$ in the $L^2(X)$ norm. Then for any $f \in L_a^2(X)$,

$$0 = \lim_{n\to\infty} \int_{\dot{X}} f(\overline{\partial}\rho_n)\,dm = \int_{\dot{X}} f\hat{h}\,dm = \int_X f\hat{h}\,dm = \int_X \hat{f} h\,dm,$$

and this completes the proof of part (iii).

Remark. Our result can be extended to higher order rational modules and Cauchy transforms as well [7].

Remark. After the writing of this note, the author received a preprint, "On the uniform approximation problem for the operator $\overline{\partial}^2$," from J. Verdera, where he obtains the following result:

Theorem 2. *Let X be a compact subset of $\mathbb{C}$ and assume that $f \in C(X)$ satisfies $\overline{\partial}^2 f = 0$ in $\dot{X}$ and*

$$|f(z) - f(w)| \le C|z - w|^\alpha, \qquad z, w \in X,$$

for some positive constants C and α. Then $f \in [\mathscr{R}(X)\overline{\mathscr{P}}_1]_u$.

Our Theorem 1(i) also follows from Theorem 2 because of the following fact (see [4] and [7]): If $f \in L_a^p(X)$, $2 < p < \infty$, then $\hat{f}$ is continuous, $\overline{\partial}^2 \hat{f} = 0$ in $\dot{X}$, and $\hat{f} \in \mathrm{Lip}(X, \alpha)$ for some $\alpha > 0$. However, the proof of Verdera's theorem is rather difficult.

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Department of Mathematics, The University of Alabama, Tuscaloosa, Alabama 35487
E-mail address: JWANG@ualvm.ua.edu