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ESSENTIAL MAPS AND MANIFOLDS

JEAN-FRANÇOIS MERTENS

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ABSTRACT. Let $(M, \partial M)$ be a compact n-manifold with boundary, orientable over a field K with characteristic q. For $f: (Y, \partial Y) \to (M, \partial M)$, with Y compact, and $(X, \partial X)$ a compact pair, $g: X \to M$, let $(P, \partial P) = \{(y, x) \in Y \times (X, \partial X) | f(y) = g(x)\}$ denote the fibered product, with p as the projection to $(X, \partial X)$. In Čech-cohomology with coefficients K, we show that if $\check{H}^n(f)$ is injective then so is $\check{H}^*(p)$—and a number of strengthenings, which point to a concept of q-essential map from one compact space to another.

Let $(M, \partial M)$ be a compact n-manifold with boundary, orientable over the ring R. For $f: (Y, \partial Y) \to (M, \partial M)$ and $(X, \partial X)$ a compact pair, $g: X \to M$, let $(P, \partial P) = \{(y, x) \in Y \times (X, \partial X) | f(y) = g(x)\}$ denote the fibered product, with p as the projection to $(X, \partial X)$. Also we fix a coefficient module G over R for homology or cohomology, and any compact space in this paper is assumed Hausdorff.

In [1] we showed that if $(X, \partial X)$ is a d-simplex, g one-to-one (this is dispensible, by the methods of the present paper), M connected, and everything is semialgebraic $(M, \partial M, Y, \partial Y, f, g, X, \text{and } \partial X)$, then letting $G_1 = f_*[H_n(Y, \partial Y; G)] \subseteq H_n(M, \partial M; G) \sim G$ and $G_2 = p_*[H_d(P, \partial P; G)] \subseteq H_d(X, \partial X; G) \sim G$, one has $G_1 \subseteq G_2$.

Here we restrict R to be a field and prove an equivalent result (i.e., showing also that cycles can be lifted through p) without any of the above restrictions. In particular, since we no longer have semialgebraicity, the formulation is rather in terms of a weakly continuous cohomology theory $\check{H}$, so Y is also assumed compact.

Theorem. *If $\check{H}^n(f)$ is one-to-one then $\check{H}^*(p)$ is also.*

Remarks. The result of [1] was a basic tool for [2]; in particular, one needed arbitrary R to show in [2] that it was a specific game theoretic property—the decomposition property—that forced one to use only fields R for defining stable sets. (Those results suggest a look at conjectures of the type: if a class of essential proper maps from locally compact spaces to Euclidean spaces is stable under products (and, say, homotopy invariant) then there is some characteristic p (zero or prime) such that those maps are all essential in the sense of

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Čech-cohomology with compact supports and with coefficients $\mathbb{Z}_p$ (with $\mathbb{Z}_0 = \mathbb{Q}$).) Given that the decomposition property basically limits consideration to fields R, the present result is the tool nneded for proving the "small worlds" property mentioned in [2], as shown in a parallel paper [3]. The situation, nevertheless, remains unsatisfactory in that we cannot simultaneously handle (even in a semialgebraic framework) arbitrary coefficient modules and arbitrary compact pairs. Here the point of getting rid of any semialgebraicity restriction is to stress the purely topological nature of such properties.

Lemma 1. *For a map $p\colon (P, \partial P) \to (X, \partial X)$ of compact pairs, $\check{H}^*(p)$ is one-to-one if and only if it is so for some vector space G' of positive dimension over the field R.*

Remark. Therefore, this property depends only on the characteristic q (zero or prime) of the field R, since G can always be viewed as a vector space over the prime field $\mathbb{Z}_q$. Thus, we can assume $G = R = \mathbb{Z}_q$.

Proof of Lemma 1. Use the universal coefficient theorem [5, VI.8.11] (torsion products are zero since R is a field).

Remark. Similarly, orientability of the manifold when the ring R is an algebra over a field K depends only on the characteristic of K and can be expressed purely in terms of Čech-cohomology as the isomorphism of $\check{H}^n(M, \partial M; K)$ with $\check{H}^0(M, \partial M; K)$.

Lemma 2. *Under the assumptions of the theorem (without the map g), there exists a triangulable, compact, orientable $\overline{n}$-manifold $\overline{M}$, a compact space $\overline{Y}$, a map $\overline{f}\colon \overline{Y} \to \overline{M}$ with $\check{H}^{\overline{n}}(\overline{f})$ one-to-one, and one-to-one maps $i\colon M \to \overline{M}$ and $j\colon Y \to \overline{Y}$ such that for every compact pair $(X, \partial X)$ and every map $g\colon X \to M$, the fibered product of $i \circ g$ and $\overline{f}$ equals (under j) that of g and f, with the same projection p.*

Proof of Lemma 2. First observe there is no loss in assuming also ∂Y compact: if it is not, let $\partial Y'$ denote its closure, $i\colon (Y, \partial Y) \subseteq (Y, \partial Y')$, $f'\colon (Y, \partial Y') \to (M, \partial M)$. Then $f = f' \circ i$ so $\check{H}^n(f) = \check{H}^n(i) \circ \check{H}^n(f')$, hence $\check{H}^n(f')$ is also one-to-one; we can work with $\partial Y'$ instead of ∂Y. By Lemma 1, we can now assume G is the prime field. We first reduce the problem to the case where $\partial M = \partial Y = \varnothing$.

If $\partial M \neq \varnothing$, glue M to a copy of itself along ∂M; thus, $(M^+, \partial M^+)$ and $(M^-, \partial M^-)$ are two copies of M and ∂M^+ and ∂M^- are identified. In this way, one obtains $\overline{M}$, which is clearly a compact manifold with subsets M^+, M^-, and ∂M. Do the same with $(Y, \partial Y)$, obtaining a compact $\overline{Y}$ that contains Y^+, Y^-, and ∂Y. Then f induces naturally a map $\overline{f}\colon \overline{Y} \to \overline{M}$. Send $(X, \partial X)$ to the corresponding subsets of $M^+ \subseteq \overline{M}$. We identify M with M^+ and Y with Y^+. It follows that the problem will be reduced to the case $\partial M = \partial Y = \varnothing$ if we prove that $\overline{M}$ is orientable and that $\check{H}^n(\overline{f})$ is one-to-one. By definition of orientability, and by [5, VI.4.8], it suffices to prove both points on each connected component separately, i.e., we can assume M and $\overline{M}$ connected.

Observe now $\overline{M}$ has a locally flat embedding into some space $\mathbb{R}^N$, with $N \geq n - 2 + \max(8, n)$, i.e., such that every point of $\overline{M}$ has a neighbourhood in $\mathbb{R}^N$ that is an $(N-n)$-ball product-bundle with its intersection with $\overline{M}$ as base. (For example, choose for each point x an open neighbourhood U_x with a homeomorphism φ_x from U_x to the open unit ball B in $\mathbb{R}^n$. Let $h(r) = \min[1, 2(1-r)^+]$, $V_x = \{y \in U_x \mid \|\varphi_x(y)\| < \frac{1}{2}\}$, $\psi_x\colon \overline{M} \to \mathbb{R}^{n+1}\colon \psi_x(y) = h(\|\varphi_x(y)\|)(1, \varphi_x(y))$ for $y \in U_x$, $\psi_x(y) = 0$ for $y \notin U_x$. ψ_x is clearly continuous and separates points of U_x as well as separating each of them from any point not in U_x. Then let $(x_i)_{i\in I}$ be a finite set such that the V_{x_i} cover $\overline{M}$: the function $\psi = (\psi_{x_i})_{i\in I}$ is the required embedding, with $N = (\#I)(n+1)$, choosing $\#I \geq 4 - \frac{1}{2}n$ to have N sufficiently large. Indeed ψ is clearly injective, and every point x of $\psi(\overline{M})$ has a neighbourhood in $\psi(\overline{M})$ of the form V_{x_i} on which the projection p to a subset of n coordinates separates points, so that the $N-n$ others are a continuous function h of those, allowing immediately the construction of the $(N-n)$-ball product-bundle as $W = \{(x', y') \in \mathbb{R}^n \times \mathbb{R}^{N-n} \mid d(x', p(x)) \leq \varepsilon,\ d(y', h(x')) \leq \varepsilon\}$, with projection p on the first factor (identified with $W \cap \overline{M}$), where $\varepsilon > 0$ is choosen sufficiently small so that $W \cap \psi(\overline{M}\backslash V_{x_i}) = \varnothing$.) Therefore, by [6, 4.5], $\overline{M}$ has a normal bundle in $\mathbb{R}^N$, i.e., an open neighbourhood O and a retraction p from O to $\overline{M}$ such that $(O, \overline{M}, \mathbb{R}^{N-n}, p)$ is a fiber bundle [5, II.7] (using also invariance of domain [5, IV.8.16] to be sure).

Consider then the open set U in $\overline{M}$ consisting of M^+ $(= M)$ together with an open collar [5, VI.2] of $(M^-, \partial M)$, with the corresponding retraction $q\colon U \to M^+$. Let $V = p^{-1}(U)$ and $r\colon V \to M\colon r = q \circ p$. Then r is a retraction from the neighbourhood V of M in $\mathbb{R}^N$ to M. Embed Y in a cube $C = [0, 1]^I$, and consider (Tietze) a continuous extension $\tilde{f}$ of f from C to $\mathbb{R}^N$. Consider the directed system of sets $C_\alpha = Y_\alpha \times [0, 1]^{I\backslash J_\alpha}$ where J_α is a finite subset of I and Y_α a polyhedron in $[0, 1]^{J_\alpha}$ such that $Y \subseteq C_\alpha$. Since the C_α decrease to Y, there exists α_0 such that $\tilde{f}(C_{\alpha_0}) \subseteq U$. Then define $f\colon C_{\alpha_0} \to M$ as $f = r \circ \tilde{f}_{|C_{\alpha_0}}$, and henceforth we consider only $\alpha \geq \alpha_0$. Given a collaring of ∂M [5, VI.6.2] one can construct a homotopy relative to ∂M between the identity on M and a map sending a neighbourhood V of ∂M into ∂M. Let $f'\colon C_{\alpha_0} \to M$ be the composition of f with this map; then f' is homotopic to f as a map from $(Y, \partial Y)$ into $(M, \partial M)$, so $\check{H}^n(f') \neq 0$; $\overline{f}'$ is homotopic to $\overline{f}$ as a map from $\overline{Y}$ to $\overline{M}$, so $\check{H}^*(\overline{f}') = \check{H}^*(\overline{f})$. It suffices thus to do the proof for the map f'; i.e., we can assume that $f\colon C_{\alpha_0} \to M$ maps a neighbourhood U of ∂Y into ∂M. This neighbourhood can, by compactness, be chosen to depend only on finitely many coordinates, say J_α $(\supseteq J_{\alpha_0})$; then let $Y_\alpha = Y_{\alpha_0} \times [0, 1]^{J_\alpha \backslash J_{\alpha_0}}$, $C_\alpha = C_{\alpha_0}$: a sufficiently fine subdivision of the triangulation of the polyhedron Y_α will be such that, for any simplex σ, letting $\hat{\sigma} = \sigma \times [0, 1]^{I\backslash J_\alpha}$, if $\hat{\sigma} \cap \partial Y \neq \varnothing$ then $f(\hat{\sigma}) \subseteq \partial M$. Denote by ∂C_α the union of all those $\hat{\sigma}$: we have $(C_\alpha, \partial C_\alpha) = (K_\alpha, \partial K_\alpha) \times [0, 1]^{I\backslash J_\alpha}$, where K_α and ∂K_α are (the space of) a simplicial complex and a (full, by one more subdivision) subcomplex, respectively. Let also $i_\alpha\colon (Y, \partial Y) \subseteq (C_\alpha, \partial C_\alpha)$ and $f_\alpha\colon (C_\alpha, \partial C_\alpha) \to (M, \partial M)$. Thus since $\check{H}^n(f) = \check{H}^n(f_\alpha \circ i_\alpha) \neq 0$, we have also $\check{H}^n(f_\alpha) \neq 0$. The system $(C_\alpha, \partial C_\alpha)$ is directed downwards by inclusion, with intersection $(Y, \partial Y)$; so the $\overline{C}_\alpha$ form a projective system with limit $\overline{Y}$ and the maps $\overline{f}_\alpha\colon \overline{C}_\alpha \to \overline{M}$ and $\overline{f}\colon \overline{Y} \to \overline{M}$ commute with this system. By the continuity property [5, VI. Example C.2, VI.6.6], it follows that $\check{H}^n(\overline{f})$ will be nonzero if we prove that $\check{H}^n(\overline{f}_\alpha)$ is nonzero for each α. Thus our problem is reduced to the case where $(Y, \partial Y) = (K, \partial K) \times [0, 1]^I$, with $(K, \partial K)$ a simplicial pair. Now let h_t: $(Y, \partial Y) \to (Y, \partial Y)$: $h_t(k, (x_i)_{i\in I}) = (k, (tx_i)_{i\in I})$, $f_t = f \circ h_t$, and π be the projection from $(Y, \partial Y)$ to $(K, \partial K)$: $f_0 = \phi \circ \pi$ and f are homotopic maps from $(Y, \partial Y)$ to $(M, \partial M)$, and similarly $\overline{f}_0$ and $\overline{f}$ are homotopic from $\overline{Y}$ to $\overline{M}$. This thus reduces the problem to the case where furthermore $f = \phi \circ \pi$, where ϕ is a map from $(K, \partial K)$ to $(M, \partial M)$. Finally, since π is a homotopy equivalence, it suffices to consider the case where $(Y, \partial Y)$ itself is a polyhedral pair. All homology and cohomology theories are now equivalent on $(Y, \partial Y)$ and on $\overline{Y}$ [5, IV.8.10, V.5] and on $(M, \partial M)$ and on $\overline{M}$ singular cohomology and Čech-cohomology coincide, respectively ([5, VI.8.8, VI.9.9, VI.1.7] and collaring). Thus we know $H^n(f) \neq 0$ and want to prove $H^n(\overline{f}) \neq 0$, all in singular cohomology.

By the universal coefficient theorem [5, V.5.3] H^n and H_n are dual finite-dimensional vector spaces; so $H^n(f)$ being nonzero is equivalent to $H_n(f) \neq 0$. Thus, let c be a simplicial n-cycle on $(Y, \partial Y)$ that is mapped to a nonzero singular cycle on $(M, \partial M)$ (using [5, IV.6.8])—thus, to a fundamental class z since R is a field and $(M, \partial M)$ is compact and connected [5, VI.3.8]. Let c^+ and c^- denote the corresponding chains on Y^+ and on Y^-; then $\overline{c} = c^+ - c^-$ is an n-cycle on $\overline{Y}$ with image $\overline{z} \in H_n(\overline{M}; R)$. $\overline{z}$ is nonzero, e.g., because its image in $H_n(\overline{M}, \overline{M} - x; R)$ equals the nonzero image of z in $H_n(M, M - x; R)$ for $x \in M^+\backslash\partial M$ (cf. [5, VI.3.8]). This yields both the orientability of $\overline{M}$ (again [5, VI.3.8]) and that $\check{H}_n(\overline{f})$ is nonzero.

Hence we can assume $\partial Y = \partial M = \varnothing$.

Recall now our previous normal bundle $(O, M, \mathbb{R}^{N-n}, p)$ for M as embedded in $\mathbb{R}^N$. In particular, M is a euclidean neighbourhood retract, and so by [6, 1.3] the bundle contains a ball-bundle, i.e., there exists a compact pair ("tubular neighbourhood") $(T, \partial T)$, with $T \subseteq O$ and $M \subseteq T\backslash\partial T$, such that the restriction of p to $(T, \partial T)$ is a ball-bundle.

Apply now [5, VI.10.15] to obtain $\theta(1)$ nonzero in $H^{N-n}(\mathbb{R}^N, \mathbb{R}^N\backslash M; R)$. We want to show that the image U of $\theta(1)$ in $(T, \partial T)$ by inclusion and excision (collaring and [5], IV.8.9) is an orientation of the bundle. It suffices to do this separately for each connected component of M. For $E \subseteq M$, let $T_E = T\cap p^{-1}(E)$, $\partial T_E = T_E \cap \partial T$, θ_E is the restriction of $\theta(1)$ to $(T_E, T_E\backslash E)$ and U_E the restriction of U—or of θ_E—to $(T_E, \partial T_E)$. For $E = \{m\}$, we will write simply T_m, etc.

Thus assume we had $U_m = 0$ for some m. Since T_m is contractible, the connecting homomorphism in the functorial exact cohomology sequences for $(T_m, \partial T_m)$ and for $(T_m, T_m\backslash\{m\})$ is an isomorphism; hence the inclusion of the first pair into the second will induce an isomorphism because the inclusion $\partial T_m \subseteq T_m\backslash\{m\}$ does, being a homotopy equivalence. Thus we have also $\theta_m = 0$.

Denote then by $\mathscr{V}$ the collection of open sets of M that are homeomorphic to $\mathbb{R}^n$ and on which the bundle is a product bundle. For any $V \in \mathscr{V}$ with $m \in V$ we would still have $\theta_V = 0$ since the inclusion $(T_m, T_m\backslash\{m\}) \subseteq (T_V, T_V\backslash V)$ is a homotopy equivalence. Hence $\mathscr{V}_0 = \{V \in \mathscr{V} | \theta_V = 0\}$ would be nonempty and any $V \in \mathscr{V}_0$ disjoint from any $V \in \mathscr{V}\backslash\mathscr{V}_0$; by connexity, $\mathscr{V}_0 = \mathscr{V}$; i.e., $\theta_V = 0 \,\forall\, V \in \mathscr{V}$. Assume now W is an open set in M with $\theta_W = 0$ and $V \in \mathscr{V}$. Let $\widetilde{W} = W \cup V$, $S = W \cap V$, and $d = N - n$, and consider the exact Mayer-Vietoris sequence [5, V.4.9]:

$$H^{d-1}(T_S, T_S\backslash S) \xrightarrow{\delta^*} H^d(T_{\widetilde{W}}, T_{\widetilde{W}}\backslash\widetilde{W}) \to H^d(T_W, T_W\backslash W) \oplus H^d(T_V, T_V\backslash V).$$

Since $S \subseteq V$, it follows that $(T_S, T_S\backslash S)$ is a product-bundle, so by Künneth's isomorphism [5, V.6.1] $H^{d-1}(T_S, T_S\backslash S)$ is zero. Hence θ_W and θ_V (which are restrictions of $\theta_{\widetilde{W}}$) being both zero imply that $\theta_{\widetilde{W}} = 0$ also. Therefore, by induction on k, we will have $\theta_W = 0$ for every union W of k elements of $\mathscr{V}$; thus, by compactness, $\theta_M = \theta(1) = 0$. This contradicts [5, VI.10.15], since $H^*(M)$ is not identically zero.

Thus our ball-bundle is orientable.

By [5, V.7.6], the fibered product $\overline{Y}$ of f and p is then an $(N-n)$-ball bundle with the projection q to Y, and say $\overline{f}$ as projection to T, and $\overline{U} = \overline{f}^*(U)$ as orientation. Further our inclusion of M in $T\backslash\partial T$ yields $Y \subseteq \overline{Y}\backslash\partial\overline{Y}$, and f is the restriction of $\overline{f}$ to Y. Write also $(\overline{M}, \partial\overline{M})$ for $(T, \partial T)$. Observe first that $(\overline{M}, \partial\overline{M})$, as a tubular neighbourhood, is clearly a compact manifold with boundary, and as embedded in $\mathbb{R}^N$, is orientable.

By the Thom isomorphism theorem [5, V.7.10] we have a commuting diagram

$$\begin{array}{ccc} H^n(M) & \xrightarrow{f^*} & H^n(Y) \\ \downarrow p_U & & \downarrow q_{\overline{U}} \\ H^N(\overline{M}, \partial M) & \xrightarrow{\overline{f}^*} & H^N(\overline{Y}, \partial\overline{Y}) \end{array}$$

with $p_U(\mu) = p^*(\mu) \cup U$, $q_{\overline{U}}(\eta) = q^*(\eta) \cup \overline{f}^*(U)$, and where the vertical arrows are isomorphisms. Therefore f^* being one-to-one would imply $\overline{f}^*$ is one-to-one. Actually, what we need is a version of this theorem in Čech-cohomology (there are trivial examples that this matters, e.g., projection on S^1 of the closure of the graph of the curve $\sin(\pi^2/\theta)$ $(0 < |\theta| \leq \pi)$): for such a version, cf. e.g., [2, part II, Appendix IV], and use [5, VI.9.5] and the five lemma for the isomorphism of singular and Čech-cohomology on M and on $(\overline{M}, \partial\overline{M})$.

Thus $\overline{f}\colon (\overline{Y}, \partial\overline{Y}) \to (\overline{M}, \partial\overline{M})$ is such that $\check{H}^N(\overline{f})$ is one-to-one, that $\overline{f}_{|Y} = f$ and that $\overline{f}(\overline{Y}\backslash Y)$ and $\partial\overline{M}$ are disjoint from M (here Y and M denote the original objects). And $(\overline{M}, \partial\overline{M})$ is a compact N-manifold with boundary, embedded in $\mathbb{R}^N$.

Observe also that connected components of M and $\overline{M}$ correspond to each other. Consider then a cube in $\mathbb{R}^N$ containing $\overline{M}$, and subdivide its triangulation until every simplex that intersects M is contained in $\overline{M}\backslash\partial\overline{M}$. Let K be the union of those simplices. Subdivide the triangulation further such that every new simplex that meets K is contained in $\overline{M}\backslash\partial\overline{M}$. Let $\overline{K}$ be the union of those simplices. A further subdivision yields a regular neighbourhood L of K in $\overline{K}$—or in the cube, with further $L \subseteq \overline{M}\backslash\partial\overline{M}$. Hence, by [4, 3.10], L is a compact PL-manifold with boundary ∂L; further $M \subseteq L\backslash\partial L$, and the connected components of L correspond bijectively to those of M and of $\overline{M}$, by construction.

Let $\partial\widetilde{M} = (M\backslash L) \cup \partial L$. We want to show that $i^*\colon \check{H}^N(\overline{M}, \partial\widetilde{M}) \to \check{H}^N(\overline{M}, \partial\overline{M})$ is one-to-one. It clearly suffices to prove this on each connected component separately. There both spaces are the underlying field R (the first by excision), so it suffices to prove $i^* \neq 0$. Including a small cube in front and our large cube at the end would otherwise yield by composition that i^* is still zero when L is a small ball included in a bigger ball $\overline{M}$, contradicting homotopy invariance. Let $j: (\overline{Y}, \partial\overline{Y}) \subseteq (\overline{Y}, \partial\widetilde{Y})$, with $\partial\widetilde{Y} = \overline{f}^{-1}(\partial\widetilde{M})$, and let $\tilde{f}: (\overline{Y}, \partial\widetilde{Y}) \to (\overline{M}, \partial\widetilde{M})$ equal $\overline{f}$. Then $\overline{f}^* \circ i^* = j^* \circ \tilde{f}^*$, so $\check{H}^N(\tilde{f})$ is also one-to-one.

Finally, let $(M', \partial M') = (L, \partial L) \subseteq (\overline{M}, \partial\widetilde{M})$ and $(Y', \partial Y') = \tilde{f}^{-1}(M', \partial M') \subseteq (\overline{Y}, \partial\widetilde{Y})$. By [5, VI.6.5], both inclusions induce isomorphisms in Čech-cohomology; so, with $f': (Y', \partial Y') \to (M', \partial M')$ equal to $\tilde{f}$, we also have $\check{H}^N(f')$ one-to-one. As before, $f'_{|Y} = f$ and $f'(Y' \setminus Y)$ and $\partial M'$ are disjoint from M. And now $(M', \partial M')$ is a compact, orientable PL-manifold with boundary.

Finally, repeat the beginning of this proof with those objects to remove the boundaries.

Proof of the theorem. By Lemmas 1 and 2, we can assume that $G = R$ is the prime field (the theorem being trivially true in the zero-dimensional case), and that M is a PL-manifold, with $\partial M = \partial Y = 0$. Further, by [5, VI.4.8], we can assume M connected. We will first prove the result in the case where g is one-to-one. X can then be viewed as a subspace of M and $(P, \partial P)$ as the inverse image by f of $(X, \partial X)$ in Y, with p the restriction of f to this space.

We first reduce this problem to the piecewise-linear case.

Fix a triangulation of M and view M as the space of this simplicial complex, thus as a subcomplex of the simplex Δ_k on the set of vertices of the triangulation. By [5, III Example A.1] M is a neighbourhood retract in Δ_k, i.e., [5, I Example C.1] there is a neighbourhood U of M in Δ_k and a retraction r from U to M. Embed Y in a cube $C = [0, 1]^I$, and consider a continuous extension $\overline{f}$ of f from C to Δ_k. For every finite subset J of I, denote by π_J the projection from C to $[0, 1]^J$, and let $C_J = \pi_J^{-1}(\pi_J(Y))$; since the C_J decrease to Y, there exists J_0 such that $\overline{f}(C_{J_0}) \subseteq U$. Define then f on C_{J_0} as $f = r \circ \overline{f}$. Now $f: C_{J_0} \to M$, and henceforth, we consider only $J \supseteq J_0$. For any J_α $(\supseteq J_0)$, let $C_\alpha = C_{J_\alpha}$, $f_\alpha = f_{|C_\alpha}$, $(P_\alpha, \partial P_\alpha) = f_\alpha^{-1}(X, \partial X)$, $\tilde{f}_\alpha = f_{|(P_\alpha, \partial P_\alpha)}$. Since $f: Y \to M$ factors into an inclusion and f_α, it follows that $\check{H}^n(f_\alpha)$ is one-to-one also. So if the theorem was established for the C_α and f_α, the weak continuity property [5, VI.6.6] will imply the result for Y since $(P_\alpha, \partial P_\alpha)$ decreases to $(P, \partial P)$. Thus we can assume $Y = Y_0 \times [0, 1]^I$, with Y_0 finite-dimensional. The same argument shows that we can replace Y_0 by a compact polyhedron containing it, and similarly that we can replace $(X, \partial X)$ by the complex of all simplices intersecting it, for some sufficiently fine subdivision of a triangulation of M. We are thus in the case where $(X, \partial X)$ is a pair of full (using one further subdivision) subcomplexes of M, and $Y = Y_0 \times [0, 1]^I$, where Y_0 is a finite simplicial complex. If the result were not true, we would have $v \in \check{H}^*(X, \partial X)$, $v \neq 0$, and $f^*(v) = 0$ in $\check{H}^*(P, \partial P)$. Use then the weak continuity property as above to find a sufficiently fine subdivision of the triangulation of M such that, denoting by $(X_1, \partial X_1)$, the simplicial neighbourhood of (i.e., the union of all simplices of the subdivision intersecting) $(X, \partial X)$, one has $v = i^* v_1$, for $v_1 \in \check{H}^*(X_1, \partial X_1)$, $i: (X, \partial X) \subseteq (X_1, \partial X_1)$, and such that $f^*(v_1) = 0$ in $\check{H}^*(P_1, \partial P_1)$, with $(P_1, \partial P_1) = f^{-1}(X_1, \partial X_1)$.

Note that f, as a continuous map to a compact metric space, depends only on a countable set I_0 of coordinates in I. Since projections on $Y_0 \times [0, 1]^{I_0}$ and on $(P, \partial P) \times [0, 1]^{I_0}$ are homotopy equivalences, we can assume I countable. Then Y is compact metric, and there exists $\varepsilon > 0$ such that the image of every ball in Y of radius $\leq \varepsilon$ is contained in some star of the triangulation of M. So there exists a finite subset, I_0 of I, and $\delta > 0$, such that for any ball C of radius $\leq \delta$ in $Y_0 \times [0, 1]^{I_0}$, $f(\pi^{-1}(C))$ is contained in some star of the triangulation of M, using π for the projection from Y to $Y_0 \times [0, 1]^{I_0}$. Since $Y_0 \times [0, 1]^{I_0}$ is a polyhedron, we can think of it as Y_0 itself; and can then subdivide its triangulation such as to have that the star of every vertex has diameter $\leq \delta$; now Y_0 is a polyhedron, I is countable, and $f(\pi^{-1}(C))$ is contained in some star of the triangulation of M for every star C in Y_0 (π projects Y to Y_0). We now use the simplicial approximation theorem. Consider the map ϕ mapping every vertex x of Y_0 to some vertex of M such that $f\ (\mathrm{star}(x) \times [0, 1]^I) \subseteq \mathrm{star}(\phi(x))$, extend ϕ by linearity to Y_0, and define $\overline{f}\colon Y \to M$ as $\phi \circ \pi$. ϕ is clearly a simplicial map, and for every $y \in Y$ the simplex spanned by $f(y)$ contains $\overline{f}(y)$.

So $(P_2, \partial P_2) = \overline{f}^{-1}(X, \partial X)$ is a pair of subcomplexes of $Y_0(\times[0, 1]^I)$ with $(P_2, \partial P_2) \subseteq (P_1, \partial P_1)$. Thus the linear homotopy connecting f and $\overline{f}$ is a homotopy both for maps from Y to M and for maps from $(P_2, \partial P_2)$ to $(X_1, \partial X_1)$. Hence our assumption on f still applies to $\overline{f}$, and the following diagram is homotopy-commutative:

$$\begin{array}{ccc} (P_2, \partial P_2) & \xrightarrow{j} & (P_1, \partial P_1) \\ \downarrow \overline{f} & & \downarrow f \\ (X, \partial X) & \xrightarrow{i} & (X_1, \partial X_1) \end{array}$$

Then $f^*(v_1) = 0$ implies $0 = (j^* \circ f^*)(v_1) = \overline{f}^*(i^*(v_1)) = \overline{f}^*(v)$; the result is also not true for the map $\overline{f} = \phi \circ \pi$.

Since π is a homotopy equivalence, it follows finally that the result is also false for the simplicial map ϕ from the polyhedron Y_0 to M: it suffices to prove the theorem when $(X, \partial X)$ is a pair of (full) subcomplexes of M, Y (the space of) a finite simplicial complex, and f a simplicial map. $(P, \partial P)$ is then also a polyhedral pair, so that now all homology and cohomology theories are equivalent.

Next we show how to reduce the problem to the case $\partial X = \varnothing$.

Since we are in the simplicial case and coefficients belong to a field, the universal coefficient theorems yield that H_q and H^q are dual finite-dimensional vector spaces, so f^* being one-to-one is equivalent to f_* being onto. We have to show that every cycle on $(X, \partial X)$ can be lifted to a cycle on $(P, \partial P)$. Let $S^1 = [-1, 1]$, where 1 and -1 are identified, and let $Y' = Y \times S^1$, $M' = M \times S^1$, and $f' = f \times \mathrm{id}_{S^1}$. For $x \in X$ let $x^+ = (x, d(x, \partial X)) \in M'$ and $x^- = (x, -d(x, \partial X))$, using for d a piecewise linear distance of diameter < 1. X^+ and X^- are the images of X in M' under those maps and note $\partial X^+ = \partial X^- = \partial X$. Also, by the Künneth formula, our assumptions are still valid for Y', M', and f'. Then if c is a cycle on $(X, \partial X)$, it can be viewed as a chain on X^+; subtracting the corresponding chain on X^- yields a cycle c' on $X' = X^+ \cup X^-$; let $\tilde{c}'$ be a cycle in P' mapped to c'. If $\tilde{c}$ denotes the chain $\tilde{c}'$ where the coefficients of all simplices that are not sent to X^+ are set to zero, then $\tilde{c}$ is a cycle on $(P^+, \partial P)$ mapped to the cycle c on $(X^+, \partial X)$. The homeomorphism setting the S^1-coordinate to zero yields the conclusion for the original sets $(X, \partial X)$ and $(P, \partial P)$.

Observe finally that it suffices to prove the theorem in the case where X is connected; otherwise, X splits into finitely many connected components whose inverse images in Y are separated, so that all homology and cohomology groups decompose into the corresponding direct sums [5, IV.4.5, V.4.10]: it suffices to have the result on each connected component separately.

Consider now $v \in H^d(X)$, $v \neq 0$. By the above-mentioned duality between homology and cohomology, there exists $z \in H_d(X)$ with $v \cap z \neq 0$ [5, V.6.19]. Now follow the proof of [5, VI.10.15]: by [5, VI.9.2], Lemma VI.10.14 still applies; use VI.9.8, VI.9.9 and VI.9.2 to find V and v', and the above-mentioned z instead of using VI.3.12. One thus obtains $u \in H^{n-d}(M, M\backslash X)$ such that $u \cup v \in H^n(M, M\backslash X)$ is nonzero.

By [5, VI.1.11, V.6.8, and the definition of the cup product before VI.10.15], one obtains the commutative diagram

$$\begin{array}{ccc} H^d(X) & \xrightarrow{\cup u} & H^n(M, M\backslash X) \\ \downarrow f^* & & \downarrow f^* \\ H^d(P) & \xrightarrow{\cup f^*(u)} & H^n(Y, Y\backslash P) \end{array}$$

Since $u \cup v \neq 0$, to prove $f^*(v) \neq 0$, it suffices to prove that the right-hand map f^* is one-to-one.

The functoriality of the cohomology sequence [5, V.4.13] yields the commutative diagram

$$\begin{array}{ccc} H^n(M, M\backslash X) & \xrightarrow{i^*} & H^n(M) \\ \downarrow f^* & & \downarrow f^* \\ H^n(Y, Y\backslash P) & \rightarrow & H^n(Y) \end{array}$$

Hence, the right-hand map f^* being one-to-one by assumption, the left-hand one will also be—thus finishing the proof—as soon as we show that i^*: $H^n(M, M\backslash X) \rightarrow H^n(M)$ is one-to-one. By the universal coefficient theorem [5, V.5.3, R is a field], i^* is the transpose of $i_*: H_n(M) \rightarrow H_n(M, M-X)$, so it suffices to prove the latter is onto. Because singular homology has compact supports [5, IV.4.6], applying the five lemma to the exact homology sequence yields that $H_n(M, M-X)$ is the direct limit of $H_n(M, M-V)$ when V varies over the open neighbourhoods of X. Thus by taking a sufficiently fine subdivision of the triangulation of M, it suffices to show that $i_*: H_n(M) \rightarrow H_n(M, M-\mathring{V})$ is onto where $\mathring{V}$ is the union of the stars of all vertices of X. Denote by V the closure of $\mathring{V}$ and let $\partial V = V - \mathring{V}$. By excision, $H_n(M, M-\mathring{V}) = H_n(V, \partial V)$ and $(V, \partial V)$ is an n-dimensional pseudomanifold with boundary [5, III. Example C] because M is an n-manifold and the subcomplex X is connected. The result is now obvious in simplicial homology theory: there are no boundaries in dimension n, the space of n-cycles on $(V, \partial V)$ is (at most) one-dimensional [5, IV. Example E.1], and for the same reason a nonzero n-cycle in M (which exists, by orientability) assigns a nonzero coefficient to each simplex, hence its restriction to $(V, \partial V)$ is a nonzero n-cycle.

This proves thus the result when g is one-to-one.

Consider now the general case, but assume first X is finite-dimensional; i.e., X can be embedded in $\mathbb{R}^k$. Denote by h such an embedding in S^k, and let $(\widetilde{Y}, \partial\widetilde{Y}) = (Y, \partial Y) \times S^k$, $(\widetilde{M}, \partial\widetilde{M}) = (M, \partial M) \times S^k$, $\tilde{f} = f \times 1_{S^k}$, and $\tilde{g} = (g, h): X \to \widetilde{M}$. Our previous result can be applied to $(\widetilde{M}, \partial\widetilde{M}, \widetilde{Y}, \partial\widetilde{Y}, \tilde{f}, X, \partial X, \tilde{g})$ so that $\tilde{p}: (\widetilde{P}, \partial\widetilde{P}) \to (X, \partial X)$ is one-to-one in cohomology. But $(\widetilde{P}, \partial\widetilde{P})$ projects (homeomorphically) to $(P, \partial P)$, say by a map q (inverse given by h), and $\tilde{p} = p \circ q$, so $\check{H}^*(\tilde{p}) = \check{H}^*(q) \circ \check{H}^*(p)$: $\check{H}^*(p)$ is also one-to-one.

Assume now $(X, \partial X) = (X_0, \partial X_0) \times [0, 1]^I$, with X_0 finite-dimensional, $g = g_0 \circ \pi$, π denoting the projection of $(X, \partial X)$ onto $(X_0, \partial X_0)$. Then also, $(P, \partial P) = (P_0, \partial P_0) \times [0, 1]^I$ and $p = p_0 \times id_{[0,1]^I}$. The previous case yields that $\check{H}^*(p_0)$ is one-to-one: hence (e.g., by the functoriality of Künneth's formula for Čech-cohomology), $\check{H}^*(p)$ is one-to-one also.

In general, view (triangulation) as before M as a subcomplex of Δ_k and X as a subset of the cube $[0, 1]^I$, with as first $(k+1)$ coordinates the compositions of g with the coordinate mappings of Δ_k denoted by I_0. Denote by π_0 the projection on $[0, 1]^{I_0}$, let $X_0 = \pi_0(X)$: g can be viewed as a (continuous) map, say g_0, from X_0 to M, so we can extend g to $\pi_0^{-1}(X_0)$ by $g = g_0 \circ \pi_0$. For any finite subset I_α of I containing I_0, let $(X_\alpha, \partial X_\alpha) = [\pi_\alpha(X, \partial X)] \times [0, 1]^{I \setminus I_\alpha}$: the $(X_\alpha, \partial X_\alpha)$ decrease to $(X, \partial X)$, the corresponding $(P_\alpha, \partial P_\alpha)$ to $(P, \partial P)$, and the maps p_α that all commute with those inverse systems satisfy for all α that $\check{H}^*(p_\alpha)$ is one-to-one, by the previous case. Thus, by the weak continuity property [5, VI.6.6], it follows that also in the limit $\check{H}^*(p)$ is one-to-one. This finishes the proof.

We obtain the following sharpening (similar to the previously mentioned application) only under an additional assumption of metrisability, which "should not" be there.

Proposition 1. *If in addition X is metrisable there exists a closed subset $\widetilde{P}$ of P such that $\check{H}^0(\tilde{p})$: $\check{H}^0(X) \to \check{H}^0(\widetilde{P})$ is an isomorphism and such that for the fibered product $\overline{p}$ of $\tilde{p}$ with any map $\tilde{g}: \widetilde{X} \to X$, where $(\widetilde{X}, \partial\widetilde{X})$ is a compact pair, one has that $\check{H}^*(\overline{p})$ is one-to-one.*

Proof. We first assume $(X, \partial X)$ an orientable d-manifold with boundary. Increasing the dimensions of Y and M, as at the end of the proof of the theorem, we can assume g is one-to-one, hence the inclusion $X \subseteq M$. For each of the finitely many connected components $(X_\alpha, \partial X_\alpha)$ of $(X, \partial X)$, let $(Y_\alpha, \partial Y_\alpha) = f^{-1}(X_\alpha, \partial X_\alpha)$, and let f_α be the restriction of f to $(Y_\alpha, \partial Y_\alpha)$ (and $(X_\alpha, \partial X_\alpha)$). By the above theorem, we know $\check{H}^d(f_\alpha)$ is one-to-one. Let $\pi = \{O_\beta | \beta \in B\}$ be an open partition of Y_α and $\partial O_\beta = O_\beta \cap \partial Y_\alpha$. Then $\check{H}^d(Y_\alpha, \partial Y_\alpha) = \Pi_\beta \check{H}^d(O_\beta, \partial O_\beta)$, by [5, VI.4.8] to be extended by exactness and five lemma to pairs. Hence, there exists $O_\pi \in \pi$ such that $\check{H}^d(f_\alpha^\pi)$ is one-to-one, letting f_α^π be the restriction of f^α to $(O_\pi, \partial O_\pi)$. Denote by $\mathscr{U}$ an ultrafilter over the partitions π, and let $V = \lim_{\mathscr{U}} O_\pi$. Clearly V is compact and connected. Further, let $\forall u \in \mathscr{U}$, $K_u = \mathrm{cl}(\bigcup_{\pi \in u} O_\pi)$, with $\partial K_u = K_u \cap \partial Y_\alpha$, $\partial V = V \cap \partial Y_\alpha$. Then $\check{H}^d(X_\alpha, \partial X_\alpha) \to \check{H}^d(K_u, \partial K_u)$ is injective for all $u \in \mathscr{U}$ since its composition with $\check{H}^d(K_u, \partial K_u) \to \check{H}^d(O_\pi, \partial O_\pi)$ is so for $\pi \in u$. Since $(V, \partial V) = \bigcap_{u \in \mathscr{U}}(K_u, \partial K_u)$, it follows then from [5, VI.6.6] that $\check{H}^d(X_\alpha, \partial X_\alpha) \to \check{H}^d(V, \partial V)$ is one-to-one. Now select such a set V (or V_α) for each X_α, and denote by $\widetilde{P}$ their union: then $\check{H}^k(\tilde{p})\colon \check{H}^k(X, \partial X) \to \check{H}^k(\widetilde{P}, \partial \widetilde{P})$ is one-to-one for $k = d$, and thus is so in all dimensions by the above theorem, and $\check{H}^0(\tilde{p})\colon \check{H}^0(X) \to \check{H}^0(\widetilde{P})$ is an isomorphism.

Now consider the general case.

Embed X in the cube $[0, 1]^{\mathbb{N}}$, as at the end of the proof of the theorem, with $g = g_0 \circ \pi_0$, where π_0 is the projection on $[0, 1]^k$, $X_0 = \pi_0(X)$, and $g_0\colon X_0 \to M$. As in the beginning of the proof of Lemma 2, g_0 can then be extended as a continuous map—still g_0—from a neighbourhood V_0 of X_0 to M. Construct now inductively a decreasing basis of neighbourhoods W_n of X in $[0, 1]^{\mathbb{N}}$, with $W_n = U_n \times [0, 1]^{\mathbb{N}\setminus I_n}$, $I_n = \{1, \dots, k+n\}$, $(U_n, \partial U_n)$ a manifold with boundary, and a pair of subcomplexes of a subdivision of $[0, 1]^{I_n}$. Note first that using the regular neighbourhood theorem [4, Proposition 3.10], every compact subset of a compact, triangulated manifold with boundary has a basis of neighbourhoods that are compact manifolds with boundary and subcomplex pairs of some subdivision of the triangulation (find first an appropriate neighbourhood that is a subcomplex in some subdivision, next use the cited theorem). Thus let $X_n = \pi_n(X)$, with π_n the projection on $[0, 1]^{I_n}$ and obtain so inductively U_n as a neighbourhood of X_n contained in V_n with $d(u, X_n) \leq \frac{1}{n}$ for all u in U_n, denoting by d the maximum distance, and let $V_{n+1} = U_n \times [0, 1]^{I_{n+1}\setminus I_n}$.

Apply then the previous case inductively, to obtain subsets P_n of the fibered product of f and g_n in $U_n \times Y$, with $g_n\colon U_n \to M$ the composition of the projection and g_0, such that, for the corresponding projection $p_n\colon P_n \to U_n$ one has $\check{H}^*(p_n)\colon \check{H}^*(U_n, \partial U_n) \to \check{H}^*(P_n, \partial P_n)$ is one-to-one and $\check{H}^0(p_n)\colon \check{H}^0(U_n) \to \check{H}^0(P_n)$ is isomorphic (to construct P_{n+1}, use p_n for f and the projection from U_{n+1} to U_n for g). Let $\widetilde{P}_n = P_n \times [0, 1]^{\mathbb{N}\setminus I_n}$, $\tilde{p}_n = p_n \times 1\colon \widetilde{P}_n \to W_n$: by homotopy equivalence, those have still the same properties. And since $\widetilde{P}_n$ and W_n decrease to $\widetilde{P}_\infty$ and X, we have indeed from [5, VI.6.6] that $\check{H}^0(\tilde{p}_\infty)\colon \check{H}^0(X) \to \check{H}^0(\widetilde{P}_\infty)$ is an isomorphism. For a compact pair $(\widetilde{X}, \partial \widetilde{X})$, with $\tilde{g}\colon \widetilde{X} \to X$, apply the previous theorem with each p_n as f and go similarly to the limit.

Remark. One way to reformulate the above is to define the following concept of "homologically onto in characteristic p":

Definition. A map $f\colon X \to Y$ (both spaces compact) is p-essential iff for every compact pair $(Z, \partial Z)$ and any map $g\colon Z \to Y$, $\check{H}^*(q)$ is one-to-one, where q is the projection on $(Z, \partial Z)$ of the fibered product Q of f and g, with $\partial Q = q^{-1}(\partial Z)$.

(Ground ring is a field of characteristic p.)

Then we obtain the following properties, either straight from the definition or from the theorem (the first of them shows that we indeed generalise exactly the usual concept where Y is a manifold).

(a) If $f\colon (Y, \partial Y) \to (M, \partial M)$ is as in the theorem, then $f\colon Y \to M$ is p-essential.

(b) If $f\colon X \to Y$ is p-essential and $\partial Y \subseteq Y$, with $\partial X = f^{-1}(\partial Y)$, then $\check{H}^*(f)\colon \check{H}^*(Y, \partial Y) \to \check{H}^*(X, \partial X)$ is one-to-one.

(c) If $f: X \to Y$ is p-essential and $g: Z \to Y$, then the projection from the fibered product of f and g to Z is p-essential.
(d) A composition of p-essential maps is still so.
(e) $f \circ g$ p-essential implies f p-essential.

In addition, the proposition suggests the conjecture that if $f: X \to Y$ is p-essential, there exists a compact $X_0 \subseteq X$, with $f_0: X_0 \to Y$ still p-essential, and $\check{H}^0(f_0)$ isomorphic. The above proof establishes this conjecture only when Y is a neighbourhood retract in the Hilbert cube_ or slightly more generally, under this assumption, any projection as sub(c) from the fibered product to Z (metrisable) will have this property.

Remark. The proposition is not fully satisfactory since, for instance in the previously mentioned application, one knows $X \backslash \partial X$ is connected and one needs a subset $\widetilde{P}$ with $\widetilde{P} \backslash \partial \widetilde{P}$ connected. (There, connexity is equivalent to variants like: every compact subset has a compact connected neighbourhood.) This we try to improve in the following. We first prove essentially another version of the above conjecture (Proposition 2), and Proposition 3 will give the results in the form that is actually needed.

Proposition 2. *Assume $f: (Y, \partial Y) \to (M, \partial M)$ is as in the theorem and that the X_n are compact metric spaces, with $g_n: X_n \to X_{n-1}$ (and $X_0 = M$), with X_n connected for $n \geq 1$. Let $h_n = g_n \circ h_{n-1}$, $h_0 = \mathbb{1}_M$. Denote by Z_n the fibered product of f and h_n, and by p_n the projection from Z_n to X_n.*

Then there exist compact connected subsets P_n of Z_n, with $(g_n \times \mathbb{1}_Y)(P_n) \subseteq P_{n-1}$, such that, denoting by $\overline{p}_n$ the restriction of p_n to P_n, for any compact pair $(X, \partial X)$, any n, and any map $g: X \to X_n$, the projection q_n from the fibered product $(Q_n, \partial Q_n)$ of $\overline{p}_n$ and g to $(X, \partial X)$ is injective in Čech-cohomology.

Further, the choice of P_n can be made completely independently of the X_i and g_i with $i > n$.

Proof. As in the proof of Proposition 1, construct first inductively an embedding of X_n in $[0, 1]^{\mathbb{N}}$ and a decreasing basis of neighbourhoods $V_n^i = U_n^i \times [0, 1]^{J_n^i}$ of X_n in $[0, 1]^{\mathbb{N}}$ such that each $(U_n^i, \partial U_n^i)$ is a connected manifold with boundary, such that g_n can be viewed as a continuous map from V_n^i to V_{n-1}^i for each i, and such that $\phi_n^i = \mathrm{Proj}_{U_{n-1}^i} \circ g_n$ is defined on U_n^i (with $U_0^i = M$).

[For this, construct first of all U_1^i, as in Proposition 1. Once all U_{n-1}^i are constructed (viewed as the spaces of simplicial complexes), rank first the compositions of g_n with the coordinate mappings of U_{n-1}^1, then with those of U_{n-1}^2, etc. Denote by ϕ_i the corresponding sequence. Intersperse the ϕ_i with whatever sequence of continuous maps from X_n to $[0, 1]$ is needed to separate points of X_n, and use the resulting sequence to define the embedding of X_n in $[0, 1]^{\mathbb{N}}$. Then extend g_n as a continuous map from $[0, 1]^{\mathbb{N}}$ (as containing X_n) to $[0, 1]^{\mathbb{N}}$ (as containing X_{n-1}), and let $W_n^i = g_n^{-1}(V_{n-1}^i)$. Then select $J_n^i = \{j_n^i, j_n^i + 1, \dots\}$, and U_n^i an appropriate (as in Proposition 1) neighbourhood of the projection of X_n on $[0, 1]^{\mathbb{N} \backslash J_n^i}$ such that in addition (by taking j_n^i sufficiently large) one has $V_n^i \subseteq W_n^i$ and that the compositions of g_n with the coordinate mappings of all U_{n-1}^k for $k \leq i$ belong to $\mathbb{N} \backslash J_n^i$.]

Then select for each i, by induction over n, using each time, e.g., Proposition 1, a compact connected subset $P_n^i (\subseteq U_n^i \times Y)$ of the fibered product of

p_{n-1}^i and ϕ_n^i, with $p_n^i\colon P_n^i \to U_n^i$ the corresponding projection (P_0^i, the fibered product of f and $\mathbb{1}_M$, is not necessarily connected) such that $\check{H}^*(p_n^i)$ is one-to-one.

Let $Q_n^i = P_n^i \times [0, 1]^{J_n^i} \subseteq V_n^i \times Y$, with $q_n^i\colon Q_n^i \to V_n^i$ the projection: $\check{H}^*(q_n^i)$ is also one-to-one and Q_n^i compact connected. Further g_n maps Q_n^i into Q_{n-1}^i. Extract now a subsequence of i's such that, for each n, the Q_n^i converge, say to P_n, in the Hausdorff topology on compact subsets. Then P_n is a compact, connected subset of $X_n \times Y$, with the projection $\overline{p}_n\colon P_n \to X_n$, such that $\check{H}^*(\overline{p}_n)$ is one-to-one, and with $g_n \times \mathbb{1}_Y$ mapping P_n into P_{n-1}. (In particular, $P_n \subseteq Z_n$, and is independent of the X_i and the g_i with $i > n$.)

Finally, given a compact pair $(X, \partial X)$ and a map $g\colon X \to X_n$, apply the theorem to the fibered product of $q_n^i\colon Q_n^i \to V_n^i$ and g, and go as above to the limit over i using weak continuity.

Proposition 3. *Assume $f\colon (Y, \partial Y) \to (M, \partial M)$ is as in the theorem and that $(X, \partial X)$ is a compact metric pair, where each compact subset of $X \backslash \partial X$ is contained in a compact connected subset. If $g\colon X \to M$ there exists a compact subset P of the fibered product of f and g such that, with p as the projection to X and $\partial P = p^{-1}(\partial X)$, one has that*

- *P is the closure of $P \backslash \partial P$ and $P \backslash \partial P$ is connected;*
- *for every compact pair $(Z, \partial Z)$, with $h\colon Z \to X$ and $h^{-1}(\partial X) \subseteq \partial Z$, the projection q from the fibered product $(Q, \partial Q)$ of p and h to $(Z, \partial Z)$ is cohomologically one-to-one.*

Proof. Let K_n be a sequence of compact, connected subsets of $X \backslash \partial X$ with $K_n \subseteq \operatorname{int}(K_{n+1})$ and $\bigcup_n K_n = X \backslash \partial X$. Let $\partial K_n = K_n \backslash \operatorname{Int}(K_n)$.

Use Proposition 2, with the inclusion maps $g_n\colon K_n \subseteq K_{n+1}$, to construct for each n

- first $P_{n,n} \subseteq \{(y, x) \in Y \times K_n | f(y) = g(x)\}$ compact connected;
- then, $\forall i < n$, $P_{n,i} \subseteq P_{n,i+1} \cap (Y \times K_i)$ compact connected such that the projections $p_{n,i}\colon P_{n,i} \to K$ are essential in the sense of Proposition 2.

In particular, letting $(X, \partial X) = (K_i, \partial K_i)$ with the identity map for g, we obtain that $q_{n,i}^*$ is one-to-one, with $q_{n,i}\colon (P_{n,i}, \partial P_{n,i}) \to (K_i, \partial K_i)$ being the projection.

Go to the limit $[(P_i, \partial P_i), q_i]$ along a subsequence of indices n along which, for all i, $P_{n,i}$ and $\partial P_{n,i}$ converge in the Hausdorff topology. Then the P_i are compact connected, with $P_i \subseteq P_{i+1} \cap \{(y, x) \in Y \times K_i | f(y) = g(x)\}$, $q_i(\partial P_i) \subseteq \partial K_i$, and q_i^* one-to-one.

Let P be the closure of $\bigcup_i P_i$ in $Y \times X$, with projection p on X, and $\partial P = p^{-1}(\partial X)$. Clearly P is a compact subset of the fibered product of f and g; since $(\bigcup_i P_i) \cap \partial P = \varnothing$, it follows also that P is the closure of $P \backslash \partial P$ and that $\bigcup_i P_i$ is dense in $P \backslash \partial P$, and hence that $P \backslash \partial P$ is connected since each P_i is so and $P_i \subseteq P_{i+1}$.

Let $\partial X_n = X \backslash \operatorname{int}(K_n)$, $\partial P^n = p^{-1}(\partial X_n)$, and $p^n\colon (P, \partial P^n) \to (X, \partial X_n)$. Then $\bigcap_n \partial X_n = \partial X$, $\bigcap_n \partial P^n = \partial P$, and, because of the excision isomorphism $(K_n, \partial K_n) \subseteq (X, \partial X_n)$ and the inclusion $(P_n, \partial P_n) \subseteq (P, \partial P^n)$, the injectivity of p_n^* follows from that of q_n^*.

Similarly, let $\partial Z_n = \partial Z \cup h^{-1}(\partial X_n)$, $\partial Q_n = q^{-1}(\partial Z_n)$. Then the ∂Z_n

and the ∂Q_n decrease to ∂Z and ∂Q. Also, by Proposition 2 and excision, let $Q_{k,n}$ denote the fibered product of $p_{k,n}: P_{k,n} \to X$ and of $h: Z \to X$, with $q_{k,n}$ as projection to X, and $\partial Q_{k,n} = q_{k,n}^{-1}(\partial Z_n)$. Then $\check{H}^*(q_{k,n}): \check{H}^*(Z, \partial Z_n) \to \check{H}^*(Q_{k,n}, \partial Q_{k,n})$ is one-to-one. The Hausdorff convergence of $P_{k,n}$ to P_n, together with weak continuity, and the inclusion of $\lim_k(Q_{k,n}, \partial Q_{k,n})$ in $(Q, \partial Q_n)$, therefore, yield the injectivity of $\check{H}^*(q_n): \check{H}^*(Z, \partial Z_n) \to \check{H}^*(Q, \partial Q_n)$, and hence the result by a last use of weak continuity.

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Center for Operations Research and Econometrics, Universite Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium

Department of Economics, SUNY, Stony-Brook, New York 11794