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MULTIPLIERS ON COMPLEMENTED BANACH ALGEBRAS

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ABSTRACT. Let A be a semisimple right complemented Banach algebra, L_A the left regular representation of A, and $M_l(A)$ the left multiplier algebra of A. In this paper we are concerned with L_A and its relationship to A and $M_l(A)$. We show that L_A is an annihilator algebra and that it is a closed ideal of $M_l(A)$. Moreover, L_A and $M_l(A)$ have the same socle. We also show that the left multiplier algebra of a minimal closed ideal of A is topologically algebra isomorphic to $L(H)$, the algebra of bounded linear operators on a Hilbert space H. Conditions are given under which L_A is right complemented.

1. INTRODUCTION

Let A be a semisimple Banach algebra. In §3 we obtain some useful results for left (right) ideals in the algebras A, L_A, and $M_l(A)$. For example, we show that every closed left ideal J of A is a left ideal of L_A. Moreover, if A contains a left approximate identity then J is also a left ideal of $M_l(A)$. A semisimple annihilator right complemented Banach algebra has this property. Section 4 is devoted to the study of L_A, where A is a semisimple right complemented Banach algebra. We show that L_A is an annihilator algebra and that it is a closed ideal of $M_l(A)$. Each minimal closed ideal of L_A is topologically algebra isomorphic to $\mathrm{LC}(H)$, the algebra of all compact linear operators on a Hilbert space H. Furthermore, L_A is right complemented if and only if $x \in \mathrm{cl}_{L_A}(xL_A)$ for all $x \in L_A$. If L_A is right complemented then it is a dual algebra.

2. PRELIMINARIES

Let A be a Banach algebra. For any subset S of A, $l_A(S)$ and $r_A(S)$ will denote, respectively, the left and right annihilators of S in A and $\mathrm{cl}_A(S)$ will denote the closure of S in A. The socle of A will be denoted by S_A. By an ideal we will always mean a two-sided ideal unless otherwise specified. We call A a modular annihilator algebra if every maximal modular left (right) ideal of A has a nonzero right (left) annihilator. A semisimple Banach algebra with dense socle is modular annihilator [15, Lemma 3.11, p. 41]. We call A an

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annihilator algebra if for every closed right ideal I, $I \neq A$, $l_A(I) \neq (0)$, and for every closed left ideal J, $J \neq A$, $r_A(J) \neq (0)$. If, in addition, $r_A(l_A(I)) = I$ and $l_A(r_A(J)) = J$, then A is called a dual algebra.

If A is a semisimple Banach algebra and K is an ideal of A, then $l_A(K) = r_A(K)$ [15, p. 37]. We denote the common value $l_A(K) = r_A(K)$ by K^a. If $S_A^a = (0)$ then every nonzero left (right) ideal of A contains a minimal idempotent ([3, p. 567] or [15, p. 34]).

Let A be a semisimple Banach algebra. A linear mapping $T: A \to A$ is called a left multiplier if $T(xy) = T(x)y$ for all x, $y \in A$. Then $M_l(A)$ be the algebra of all left multipliers on A. Since every left multiplier on A is continuous [6], $M_l(A)$ is a Banach algebra under the operator bound norm. For each $a \in A$, let L_a be the operator on A given by $L_a(x) = ax$, $x \in A$. Then $L_a \in M_l(A)$, for all $a \in A$, and the mapping $a \to L_a$ is a norm-decreasing algebra isomorphism of A into $M_l(A)$ and embeds A as a left ideal of $M_l(A)$, [12, 14]. Let L_A be the closure of $\{L_a: a \in A\}$ in $M_l(A)$. We call L_A the *left regular representation of* A. In what follows we will identify A as a left ideal of $M_l(A)$ and as a dense left ideal of L_A. In the terminology of [7], A is an abstract Segal algebra in L_A. It is shown in [14] that every subalgebra B of $M_l(A)$ such that $A \subset B$ is semisimple. Thus, in particular, L_A is semisimple. (See also [12].)

All Banach algebras considered in this paper are over the complex field. When necessary we will denote the norm in a Banach algebra A by $\|\cdot\|_A$. This will occur when two or more Banach algebras are involved at the same time. Otherwise the norm in A will be denoted simply as $\|\cdot\|$.

Let X be a Banach space. Then $L(X)$ will denote the algebra of all bounded linear operators on X and $\mathrm{LC}(X)$ the subalgebra of all compact linear operators on X. If S is a subspace of X and $T \in L(X)$, then $T|S$ will denote the restriction of T to S.

Let A be a Banach algebra and let L_r be the set of all closed right ideals in A. We say that A is *right complemented* (r.c.) if there exists a mapping $p: I \to I^p$ of L_r into itself (called a right complementor) having the following properties:

(C_1) $R \cap R^p = (0)$ $(R \in L_r)$;
(C_2) $R + R^p = A$ $(R \in L_r)$;
(C_3) $(R^p)^p = R$ $(R \in L)r)$;
(C_4) if $R_1 \subseteq R_2$ then $R_2^p \subseteq R_1^p$ $(R_1, R_2 \in L_r)$.

A semisimple r.c. Banach algebra A has dense socle [11, Lemma 5, p. 655] and, for every $x \in A$, $x \in \mathrm{cl}_A(xA)$ [1, Lemma 3, p. 39].

We put together several useful results in the following lemma.

Lemma 2.1. *Let A be a semisimple Banach algebra that is a dense subalgebra of a semisimple Banach algebra B. Then the following statements hold.*

(i) *If S_A is a dense ideal of B then $S_A = S_B$.*
(ii) *If A is an annihilator algebra then S_A is a dense ideal of B.*
(iii) *Assume that S_A is dense in A. If B is an annihilator algebra and S_A is an ideal of B, then A is an annihilator algebra.*

Proof. (i) This is contained in the proof of [13, Lemma 4.1, p. 262].
(ii) This is proved in [14]. (See [14, Corollary 4.4, p. 128].)

(iii) Suppose that B is an annihilator algebra and that S_A is an ideal of B. Then, by [4, Corollary, p. 1036], the identity embedding of A into B is continuous. Therefore S_A is dense in B and so $S_A = S_B$ by (i). Thus the norms $\|\cdot\|_A$ and $\|\cdot\|_B$ are equivalent on every minimal right ideal of B. Since B is an annihilator algebra, [13, Corollary 6.11, p. 276] implies that every minimal right ideal of B is a reflexive Banach space and every minimal idempotent of B is full. The argument above shows that these properties also hold for minimal right ideals and minimal idempotents in A. Since S_A is dense in A, we may now apply [13, Corollary 6.11, p. 276] to A to show that A is an annihilator algebra.

3. Ideals in A, L_A, and $M_l(A)$

Proposition 3.1. *Let A be a semisimple Banach algebra and let B be any subalgebra of $M_l(A)$ such that $A \subset B$. If R is any nonzero right ideal of B then $R \cap A \neq (0)$.*

Proof. Suppose $R \cap A = (0)$. Since A is a left ideal of B, we have $RA \subset R \cap A$ so that $RA = (0)$. Thus if $T \in R$ then $TL_x = 0$ for all $x \in A$. Therefore $TL_x(y) = T(xy) = T(x)y = 0$ for all x, $y \in A$, which shows that $T(x) \in l_A(A)$ for all $x \in A$. The semisimplicity of A implies that $T(x) = 0$ for all $x \in A$. Thus $T = 0$ and so $R = (0)$, a contradiction. Hence $R \cap A \neq (0)$.

Proposition 3.2. *Let A be a semisimple Banach algebra, and let B be a subalgebra of $M_l(A)$ such that $A \subset B$. Then S_A is a left ideal of B and $S_A \subseteq S_B$.*

Proof. Let e be a minimal idempotent of A. Since A is a left ideal of B, $Be \subset A$ and therefore $Be = Bee \subset Ae$. As $Ae \subset Be$, we get $Be = Ae$. Hence $S_A \subseteq S_B$.

In a similar way we can show that if $L_A \subset B$ then $S_{L_A} \subseteq S_B$.

Proposition 3.3. *Let A be a semisimple Banach algebra, and let B be a subalgebra of $M_l(A)$ such that $A \subset B$. If $S_A^a = (0)$ then $S_B^a = (0)$ and every nonzero right ideal of B contains a minimal idempotent of A.*

Proof. Suppose that $S_A^a = (0)$. If $S_B^a \neq (0)$ then, by Proposition 3.1, $S_B^a \cap A \neq (0)$ and so contains a minimal idempotent e of A. Since $S_A \subseteq S_B$, this means that $e \in S_B^a \cap S_B$ so that $e = e^2 = 0$, a contradiction. Therefore $S_B^a = (0)$.

Corollary 3.4. *Let A be a semisimple modular annihilator Banach algebra, and let B be any subalgebra of $M_l(A)$ such that $A \subset B$. Then every nonzero right ideal of B contains a minimal idempotent of A.*

Proof. This is an immediate consequence of Proposition 3.3 since $S_A^a = (0)$ [3, Theorem 4.2, p. 269].

Proposition 3.5. *Let A be a semisimple Banach algebra. Then every closed left ideal of A is a left ideal of L_A.*

Proof. Let J be a closed left ideal of A and let $\mathbf{J} = \{L_a : A \in J\}$. We show that $\mathbf{J}$ is a left ideal of L_A. Let $T \in L_A$ and let $\{a_n\}$ be a sequence in A such that $L_{a_n} \to T$. Let $y \in J$. Then $L_{a_n}(y) = a_n y \in J$ and $L_{a_n}(y) \to T(y)$. As J is closed, $T(y) \in J$. But $TL_y(x) = T(yx) = T(y)x = L_{T(y)}(x)$, for all $x \in A$, implies that $TL_y = L_{T(y)}$. Hence $TL_y \in \mathbf{J}$ and so $\mathbf{J}$ is a left ideal of L_A. Identifying J with $\mathbf{J}$ completes the proof.

Corollary 3.6. *Let A be a semisimple Banach algebra. Then for every closed ideal I of A, $\mathrm{cl}_{L_A}(I)$ is a closed ideal of L_A.*

Proof. Let $\mathbf{I} = \{L_a : a \in I\}$. In view of Proposition 3.5 we need only show that $\mathrm{cl}_{L_A}(\mathbf{I})$ is a right ideal of L_A. Let $T \in \mathrm{cl}_{L_A}(\mathbf{I})$ and $S \in L_A$. Let $\{a_n\}$ be a sequence in I such that $L_{a_n} \to T$ and $\{b_n\}$ be a sequence in A such that $L_{b_n} \to S$. Since $L_{a_n b_n} \in \mathbf{I}$ for all n and $TS = \lim_{n\to\infty}(L_{a_n} L_{b_n}) = \lim_{n\to\infty} L_{a_n b_n}$, we see that $TS \in \mathrm{cl}_{L_A}(\mathbf{I})$. Identifying I with $\mathbf{I}$ completes the proof.

If A has a left approximate identity (not necessarily bounded) then Proposition 3.5 takes the following more general form.

Proposition 3.7. *Let A be a semisimple Banach algebra with a left approximate identity. Then every closed left ideal of A is a left ideal of $M_l(A)$.*

Proof. Let $\{u_\gamma\}$ be a left approximate identity in A, and let J be a closed left ideal of A. Let $a \in J$ and $T \in M_l(A)$. Since $\|a - u_\gamma a\| \to 0$ and T is continuous, we have $\|T(a) - T(u_\gamma a)\| \to 0$. That is, $T(u_\gamma a) = T(u_\gamma)a \to T(a)$. Since J is closed and $T(u_\gamma)a \in J$ for all γ, it follows that $T(a) \in J$. We can now apply the argument given in the proof of Proposition 3.5 to show that J is a left ideal of $M_l(A)$.

Corollary 3.8. *Let A be a semisimple annihilator right complemented Banach algebra. Then every closed left ideal of A is a left ideal of $M_l(A)$.*

Proof. By [12, Theorem 3.7, p. 75], A has a left approximate identity that is bounded in the norm of L_A. Application of Proposition 3.7 completes the proof.

4. Main results

In this section we study L_A where A is a semisimple r.c. Banach algebra with a right complementor p. Since A is semisimple, so is L_A.

Theorem 4.1. *Let A be a semisimple right complemented Banach algebra. Then L_A is a semisimple annihilator algebra.*

Proof. Let K be a minimal closed ideal of A. Then K is a topologically simple and semisimple r.c. Banach algebra [11, Lemma 1, p. 652]. Let e be a minimal idempotent contained in K. Then $I = Ae$ is a minimal left ideal of K (and A) and so is a Hilbert space under an equivalent norm. If I is finite-dimensional this is clear, and if I is infinite-dimensional this follows from [11, Theorem 5, p. 652]. (See also [1, p. 40].) Denote this Hilbert space by H. Let $\varphi : a \to T_a$ be the representation of K on H corresponding to the left regular representation of K on I, i.e., $T_a(x) = ax$ for all $x \in I$. Then φ is a faithful, continuous, and strictly dense representation of K on H. Hence if K is finite-dimensional then $\varphi(K) = L(H)$. If K is infinite-dimensional then it follows from [1, Theorem 1, p. 40] that $ET_a \in \varphi(K)$, for all orthogonal projections E on H and all $a \in K$. Thus, by [9, Theorem 1, p. 454], $\varphi(K)$ is a left ideal of $L(H)$. (See also [2, p. 391].) Since the socle of $\varphi(K)$ is dense in $\mathrm{LC}(H)$, the algebra of all compact linear operators on H, it follows that $\varphi(K)$ is a dense left ideal of $\mathrm{LC}(H)$. Thus $\varphi(K)$ is an abstract Segal algebra in $\mathrm{LC}(H)$ [7, Proposition 1.6, p. 299]. Therefore, by [12, Proposition 2.2, p. 73], the left regular representation L_K of K is topologically algebra isomorphic

to $\mathrm{LC}(H)$. For each $a \in K$, let L_a^K be the left multiplication by a on K, i.e., $L_a^K(x) = ax$ for all $x \in K$. Then $L_a^K \in L_K$ and $L_a^K = L_a|K$. Since $A = K \oplus K^p$, there exists a constant $D_K > 0$ such that if $x \in A$ and we write $x = x_1 + x_2$, $x_1 \in K$ and $x_2 \in K^p$, then $\|x_i\| \leq D_K\|x\|$ for $i = 1, 2$.

For $T \in M_l(K)$, let T' be the mapping on A given as follows: For $x \in A$, $x = x_1 + x_2$, $x_1 \in K$, and $x_2 \in K^p$, define $T'(x) = T(x_1)$. Then T' is linear and $\|T'\| \leq D_K\|T\|$, where $\|T\|$ denotes the norm of T over K. Clearly $\|T\| \leq \|T'\|$. Moreover, using the fact that $K^p = l_A(K) = r_A(K)$ [11, Lemma 1, p. 652] and $K \oplus K^p = A$, it is easy to see that $T' \in M_l(A)$. We have $T'|K = T$ and $T'(K^p) = (0)$. Also if T_1, $T_2 \in M_l(K)$ then $(T_1T_2)' = T_1'T_2'$. Hence the mapping $\rho_K\colon T \to T'$ is a bicontinuous algebra isomorphism of $M_l(K)$ into $M_l(A)$ such that $\rho_K(L_a^K) = L_a$ for all $a \in K$. Thus, in particular, $\rho_K(L_K)$ is a closed subalgebra of $M_l(A)$. Since L_K is the closure of $\{L_a^K\colon a \in K\}$ in $M_l(K)$, it follows that $\rho_K(L_K)$ is the closure of $\{L_a\colon a \in K\}$ in $M_l(A)$. Therefore $\rho_K(L_K) \subset L_A$ and $\rho_K(L_K) = \mathrm{cl}_{L_A}(\{L_a\colon a \in K\})$. For convenience of notation, let $\mathbf{K} = \rho_K(L_K)$. Identifying A as a subalgebra of L_A, we get $\mathbf{K} = \mathrm{cl}_{L_A}(K)$. By Corollary 3.6, $\mathbf{K}$ is a closed ideal of L_A. Since $\mathbf{K}$ is topologically algebra isomorphic of $\mathrm{LC}(H)$, $\mathbf{K}$ is an annihilator algebra. Clearly $\mathbf{K}$ is a minimal closed ideal of L_A.

Let $\{K_\alpha\colon \alpha \in \Omega\}$ be the family of all distinct minimal closed ideals in A. By the argument above, for each $\alpha \in \Omega$, $\mathbf{K}_\alpha = \mathrm{cl}_{L_A}(K_\alpha)$ is a minimal closed ideal of L_A and is an annihilator algebra. Since $\sum_\alpha K_\alpha$ is dense in A, it follows that $\sum_\alpha \mathbf{K}_\alpha$ is dense in L_A. Therefore, by [10, Theorem (2.8.29), p. 106], L_A is an annihilator algebra.

Corollary 4.2. *Let A be a semisimple right complemented Banach algebra. Then the mapping $K \to \mathrm{cl}_{L_A}(K)$ is a one-to-one correspondence between the set of all minimal closed ideals of A and the set of all minimal closed ideals of L_A, and $K = \mathrm{cl}_{L_A}(K) \cap A$. Moreover, every minimal closed ideal of L_A is topologically algebra isomorphic to $\mathrm{LC}(H)$, the algebra of all compact linear operators on a Hilbert space H.*

Proof. We only need to verify that $K = \mathrm{cl}_{L_A}(K) \cap A$, where K is a minimal closed ideal of A, the rest is clear from the proof above. Let $\mathbf{K} = \mathrm{cl}_{L_A}(K)$ and let $K' = \mathbf{K} \cap A$. Then K' is a closed ideal of A and, therefore, is a semisimple right complemented Banach algebra in its own right [11, Lemma 1, p. 652]. Hence if $K \neq K'$ then there exists a nonzero closed ideal J in K' such that $K \oplus J = K'$. Thus, in particular, $KJ = (0)$. Since $\mathbf{K} = \mathrm{cl}_{L_A}(K)$, it follows that $\mathbf{K}J = (0)$. This is impossible since $J \subset \mathbf{K}$ and $\mathbf{K}$ is semisimple. Hence $J = (0)$ and so $K = \mathbf{K} \cap A$.

Corollary 4.3. *A semisimple right complemented Banach algebra with a bounded right approximate identity is an annihilator algebra.*

Proof. Since A has a bounded right approximate identity, the norms $\|\cdot\|_A$ and $\|\cdot\|_{L_A}$ are equivalent on A. Hence the mapping $a \to L_a$ takes A onto L_A.

Corollary 4.4. *Let A be a semisimple right complemented Banach algebra. Then A is an annihilator algebra if and only if S_A is an ideal of L_A. In this case we have $S_A = S_{L_A}$.*

Proof. This follows immediately from Lemma 2.1 and Theorem 4.1.

Theorem 4.5. *Let A be a semisimple right complemented Banach algebra and let K be a minimal closed ideal of A. Then $M_l(K)$ is topologically algebra isomorphic to $L(H)$ for some Hilbert space H.*

Proof. By the proof of Theorem 4.1, there exists a Hilbert space H such that K can be continuously embedded as a dense left ideal of $\mathrm{LC}(H)$ and L_K is topologically algebra isomorphic to $\mathrm{LC}(H)$. Hence $M_l(L_K)$ is topologically algebra isomorphic to $M_l(\mathrm{LC}(H)$. In what follows we will identity K as a dense left ideal of $\mathrm{LC}(H)$. Now, by [12, Proposition 3.1, p. 74], every $S \in M_l(K)$ has a unique extension S' to L_K, $S' \in M_l(L_K)$, and $\|S'\| \leq \|S\|$. Thus the mapping $S \to S'$ is a continuous algebra isomorphism of $M_l(K)$ into $M_l(L_K)$. By [8, Lemma 2.1, p. 506], $M_l(\mathrm{LC}(H))$ is isometrically algebra isomorphic to $L(H)$. Therefore, $M_l(L_K)$ is topologically algebra isomorphic to $L(H)$ and so there exists a continuous algebra isomorphism σ of $M_l(K)$ into $L(H)$. Since K is a left ideal of $L(H)$, each $T \in L(H)$ gives rise to the left multiplier $S = L_T|K \in M_l(K)$. Hence σ is onto and so $M_l(K)$ is topologically algebra isomorphic to $L(H)$. Thus the socle of $M_l(K)$ is mapped by σ onto the socle of $L(H)$, and the socle of $L(H)$ is equal to the socle of $\mathrm{LC}(H)$. As $\mathrm{LC}(H)$ is topologically algebra isomorphic to $L_K \subset M_l(K)$, it follows that the socle of $M_l(K)$ is equal to the socle of L_K.

Corollary 4.6. *Let A be a semisimple right complemented Banach algebra and let K be a minimal closed ideal of A. Then $M_l(K)$ is topologically algebra isomorphic to $M_l(L_K)$. Moreover, the socle of $M_l(K)$ is equal to the socle of L_K so that, in particular, L_K is a closed ideal of $M_l(K)$.*

We will show below that also L_A is an ideal of $M_l(A)$. We observe that if I is a closed ideal of A, then $T(I) \subseteq I$ for all $T \in M_l(A)$. In fact, I is a semisimple r.c. Banach algebra in its own right [11, Lemma 1, p. 652] and therefore has dense socle S_I. Since every minimal left ideal J contained in I is of the form $J = Ae$, $e^2 = e$, we get $T(J) \subseteq J$. Thus $T(S_I) \subseteq S_I$ and the continuity of T implies that $T(I) \subseteq I$. Thus, in particular, $T|I \in M_l(I)$ for all $T \in M_l(A)$.

Let K be a minimal closed ideal of A. Since $T|K \in M_l(K)$ for all $T \in M_l(A)$, we see that $M_K = \{T|K\colon T \in M_l(A)\} \subseteq M_l(K)$. On the other hand, since for each $T \in M_l(K)$, $T' = \rho_K(T) \in M_l(A)$ and $T = T'|K$ (see the proof of Theorem 4.1), we have $M_l(K) \subseteq M_K$. Hence $M_K = M_l(K)$. We know that $\rho_K(M_l(K))$ is a closed subalgebra of $M_l(A)$. We claim that it is also an ideal of $M_l(A)$. Note that $\rho_K(M_l(K)) = \{T'\colon T \in M_l(K)\}$. Let $T \in M_l(K)$, $S \in M_l(A)$, and $x \in A$. Write $x = x_1 + x_2$ with $x_1 \in K$ and $x_2 \in K^p$. Then $(T'S)(x) = T'(S(x_1 + x_2)) = T'(S(x_1) + S(x_2)) = T(S(x_1)) = (T(S|K))(x_1) = (T(S|K))'(x)$. Hence $T'S = (T(S|K))' \in \rho_K(M_l(K))$. Likewise $ST' \in \rho_K(M_l(K))$. This verifies our claim.

Theorem 4.7. *Let A be a semisimple right complemented Banach algebra. Then L_A is a closed ideal of $M_l(A)$.*

Proof. To simplify notation, let $B = M_l(A)$. Let $A_L = \{L_a\colon a \in A\}$ and let e be a minimal idempotent in B. By Corollary 3.4, eB contains a minimal idempotent f of A_L. We have $f = L_g$, for some minimal idempotent $g \in A$. Since $S_{L_A} \subseteq S_B$, it follows that f is also a minimal idempotent of B and $eB = fB$. Let $I = \mathrm{cl}_B(BeB) = \mathrm{cl}_B(BfB)$ and $K = \mathrm{cl}_A(AgA)$. Then I (resp.

K) is a minimal closed ideal of B (resp. A). Since $f \in \rho_K(M_l(K)) \cap I$, it follows that $\rho_K(M_l(K)) \cap I \neq (0)$ and therefore, by the minimality of I, $\rho_K(M_l(K)) \cap I = I$. This shows that $e \in \rho_K(M_l(K))$. Now $L_K \subset M_l(K)$ and, by Corollary 4.6, $S_{L_K} = S_{M_l(K)}$. Hence $e \in \rho_K(L_K) \subset L_A$ and so $Be \subset S_{L_A}$. Thus $S_B \subseteq S_{L_A}$. As $S_{L_A} \subseteq S_B$, we obtain $S_B = S_{L_A}$. Since S_{L_A} is dense in L_A and $S_{L_A} = S_B$ is an ideal of B, it follows that L_A is a closed ideal of B.

Corollary 4.8. *Let A be a semisimple right complemented Banach algebra. Then $S_{L_A} = S_{M_l(A)}$.*

Corollary 4.9. *Let A be a semisimple right complemented Banach algebra. Then A is an annihilator algebra if and only if $S_A = S_{M_l(A)}$.*

Proof. This follows immediately from Corollaries 4.4 and 4.8.

Theorem 4.10. *Let A be a semisimple right complemented Banach algebra. Then L_A is right complemented if and only if $x \in \mathrm{cl}_{L_A}(xL_A)$ for all $x \in L_A$.*

Proof. To simplify notation, let $B = L_A$, and let p be the right complementor on A. If B is right complemented then, by [1, Lemma 3, p. 39], $x \in \mathrm{cl}_B(xB)$ for all $x \in B$; i.e., B has approximate right units [7, p. 299]. Conversely suppose that B has approximate right units. Let L_r ($\mathbf{L}_r$) be the set of all closed right ideals in $A(B)$. Since A also has approximate right units and A is an abstract Segal algebra in B, by [7, Theorem 2.3, 299], the mapping $I \to \mathrm{cl}_B(I)$ is a bijection of L_r onto $\mathbf{L}_r$ and $\mathrm{cl}_B(I) \cap A = I$. For $R \in \mathbf{L}_r$, let $R^q = \mathrm{cl}_B([R \cap A]^p)$. We claim that q is a right complementor on B. To see that (C_1) is satisfied, let $I = R \cap R^q$. Then $I \cap A$ is a closed right ideal of A, $I \cap A \subset R \cap A$, and $I \cap A \subset R^q \cap A = [R \cap A]^p$. Hence $I \cap A \subset (R \cap A) \cap [R \cap A]^p = (0)$. Therefore $I = (0)$. Property (C_3) also holds for q since

$$\begin{aligned}(R^q)^q &= \mathrm{cl}_B([\mathrm{cl}_B([R \cap A]^p) \cap A]^p) = \mathrm{cl}_B(([R \cap A]^p)^p) \\ &= \mathrm{cl}_B(R \cap A) = R.\end{aligned}$$

Moreover, if R_1, $R_2 \in \mathbf{L}_r$, $R_1 \subseteq R_2$, then $R_2^q \subseteq R_1^q$ since $R_1 \cap A \subseteq R_2 \cap A$ and $[R_2 \cap A]^p \subseteq [R_1 \cap A]^p$. Therefore q satisfies (C_4). Since the norm $\|\cdot\|_B$ in B has the property that $\|a\|_B = \sup\{\|ax\|_A : \|x\|_A \leq 1,\ x \in A\}$ for all $a \in A$, we can apply verbatim the argument in the proof of [13, Theorem 5.2(i), p. 265] to show that q satisfies (C_2). Therefore q is a right complementor on $B = L_A$ and this completes the proof.

Theorem 4.11. *Let A be a semisimple right complemented Banach algebra. If L_A is right complemented then L_A is a dual algebra.*

Proof. Suppose that L_A is right complemented. Then $x \in \mathrm{cl}_{L_A}(xL_A)$ for all $x \in L_A$. Since L_A is an annihilator algebra, by [12, Theorem 3.6, p. 75], L_A has a quasi-bounded left approximate identity so that $x \in \mathrm{cl}_{L_A}(L_A x)$ for all $x \in L_A$. Thus $x \in \mathrm{cl}_{L_A}(xL_A) \cap \mathrm{cl}_{L_A}(L_A x)$ for all $x \in L_A$. Therefore, by the proof of [10, Theorem (2.8.27), p. 104], L_A is dual.

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