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COMPACT AND WEAKLY COMPACT HOMOMORPHISMS BETWEEN ALGEBRAS OF DIFFERENTIABLE FUNCTIONS

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ABSTRACT. It is proved that the weakly compact continuous homomorphisms between algebras of k-times continuously differentiable functions on Banach spaces are induced by constant mappings.

Recently many authors have studied compact and weakly compact homomorphisms between function algebras. Lindström and Llavona [5] treat weakly compact continuous homomorphisms between algebras of the type $C(T)$, where T is a completely regular Hausdorff space; Feldman [1] deals with compact composition operators on some function spaces, and Kamowitz [4] describes the compact endomorphisms of $C^1[0, 1]$.

Llavona asked whether the results in [5] are valid in the case of algebras of differentiable functions on Banach spaces. The purpose of this note is to give an affirmative answer to this question by proving that weakly compact continuous homomorphisms between algebras of differentiable functions are induced by constant mappings. The difficulty we face is that in [5] the existence of continuous functions separating points and closed sets plays an essential role, while in the differentiable case these functions do not exist in general. In this note we deal with Fréchet differentiability, but our results are also valid for Hadamard differentiable functions.

$\mathbb{R}$ denotes the real field and $\mathbb{N}$ the nonnegative integers. E and F are real Banach spaces, E^* the topological dual of E, and $C^k(E)$ the space of all real-valued k-times continuously Fréchet differentiable functions on E. Two topologies are *natural* on $C^k(E)$ (see e.g. [6]):

(a) The compact open topology of order k given by the seminorms:

$$p_L(f) = \sup\{\|d^j f(x)\| : x \in L;\ 0 \leq j \leq k\},$$

where $d^0 f = f$, $d^j f$ is the jth derivative of f, and L runs through the compact subsets of E. We recall that $d^j f$ is a continuous mapping from E

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into the space $\mathscr{P}({}^jE)$ of real-valued j-homogeneous continuous polynomials on E.

(b) The compact-compact topology of order k, given by the seminorms:

$$q_L(f) = \sup\{|f(x)|, |d^j f(x)(y)| : x, y \in L; \ 1 \le j \le k\},$$

where L is allowed to range on the compact subsets of E.

Both topologies coincide only when E is finite dimensional.

When we say that an algebra homomorphism $A: C^k(E) \to C^k(F)$ is continuous, we understand that both $C^k(E)$ and $C^k(F)$ are endowed with one of the *natural* topologies. In [3] it is proved that any nonzero continuous homomorphisms A as above is of the form $Af := f \circ \varphi$ $(f \in C^k(E))$, where $\varphi: F \to E$ satisfies $\phi \circ \varphi \in C^k(F)$ for each $\phi \in E^*$ (and so, if $\dim(E) < \infty$, then $\varphi \in C^k(F)$). For completeness, we sketch the proof. First, by standard arguments (see e.g. [2]), it can be shown that, for any nonzero continuous homomorphism $\Phi: C^k(E) \to \mathbb{R}$, there is a unique $x \in E$ such that $\Phi(f) = f(x)$ for every $f \in C^k(E)$. Now, let $\delta_y: C^k(F) \to \mathbb{R}$ be the evaluation map at $y \in F$. Then $\delta_y \circ A$ is a nonzero continuous homomorphism from $C^k(E)$ to $\mathbb{R}$, so there is a unique $x \in E$ such that $Af(y) = \delta_y \circ Af = f(x)$ for every $f \in C^k(E)$. To finish the proof, it is enough to let $\varphi: F \to E$ be the map taking y to x. For a wide class of Banach spaces E (including all the separable spaces and their duals), any algebra homomorphism $A: C^k(E) \to C^k(F)$ is automatically continuous [3].

Following [5] we say that a homomorphism $A: C^k(E) \to C^k(F)$ is (weakly) compact if it maps bounded subsets of $C^k(E)$ into relatively (weakly) compact subsets of $C^k(F)$.

We first prove the result in the case $E = F = \mathbb{R}$, and then derive the general case from it.

1. **Proposition.** *Let $A: C^k(\mathbb{R}) \to C^k(\mathbb{R})$ be a nonzero algebra homomorphism, for some $k \in \mathbb{N}\backslash\{0\}$, and let $\varphi \in C^k(\mathbb{R})$ be its inducing function. Then the following assertions are equivalent*:

(a) *φ is constant*;
(b) *A has one-dimensional rank*;
(c) *A is compact*;
(d) *A is weakly compact.*

Proof. (a) $\Rightarrow$ (b). If $\varphi(y) = x_0$ for every $y \in \mathbb{R}$, then $Af = f(x_0) \cdot \mathbf{1}$ $(f \in C^k(\mathbb{R}))$, where $\mathbf{1}$ is the constant function with value 1.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d). They are obvious.

(d) $\Rightarrow$ (a). If φ is not constant, then we may assume there exist $a < b$ such that $\varphi(a) < \varphi(b)$, φ is increasing in $[a, b]$ and $\varphi'(x) \ge m > 0$ for each $x \in [a, b]$.

Let $\psi := \varphi|_{[a,b]}$, and let $B: C^k(\mathbb{R}) \to C^k[a, b]$ denote the homomorphism given by $Bf = f \circ \psi$ $(f \in C^k(\mathbb{R}))$.

Choose a sequence (x_n) with:

$$\psi(a) < x_1 < \cdots < x_n < \cdots < \psi(b),$$

and $|x_{n+1} - x_n| \le h < 1$ for every $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we can find a function $f_n \in C^k(\mathbb{R})$ supported by (x_n, x_{n+1}), with $\|f_n^{(k)}\|_\infty = 1$. Therefore, we have $\|f_n^{(k-j)}\|_\infty < h^j \quad (1 \le j \le k)$.

If $y_i = \psi^{-1}(x_i)$, for every $i \in \mathbb{N}$, we then have:

$$a < y_1 < \cdots < y_n < \cdots < b,$$

and $f_n \circ \psi$ is supported by $(y_n, y_{n+1}) \quad (n \in \mathbb{N})$.

The sequence (f_n) is equivalent to the unit vector basis of c_0. Indeed, if (a_n) is a finite sequence of real numbers and $\|\cdot\|$ denotes the norm in the Banach space $C^k[\psi(a), \psi(b)]$, then since (f_n) have disjoint supports, we have:

$$\left\|\sum a_n f_n\right\| = \sup\left\{\left|\sum a_n f_n^{(j)}(t)\right| : 0 \le j \le k;\ t \in [\psi(a),\ \psi(b)]\right\} = \sup_n |a_n|.$$

The sequence (Bf_n) is also equivalent to the basis of c_0. Again, for every finite sequence (a_n) of real numbers, since $(f_n \circ \psi)$ have disjoint supports, we have:

$$\begin{aligned}
\left\|\sum a_n B f_n\right\| &= \left\|\sum a_n f_n \circ \psi\right\| \\
&= \sup\left\{\left|\sum a_n (f_n \circ \psi)^{(j)}(t)\right| : 0 \le j \le k;\ a \le t \le b\right\} \\
&= \sup_n \sup\{|a_n| \cdot |(f_n \circ \psi)^{(j)}(t)| : 0 \le j \le k;\ y_n < t \le y_{n+1}\} \\
&= \sup_n |a_n| \cdot \sup\{|(f_n \circ \psi)^{(j)}(t)| : 0 \le j \le k;\ a \le t \le b\} \\
&\le M \cdot \sup_n |a_n|
\end{aligned}$$

for some $M > 0$. Moreover,

$$\begin{aligned}
&\sup\{|(f_n \circ \psi)^{(j)}(t)| : 0 \le j \le k;\ a \le t \le b\} \\
&\quad \ge \sup\left\{ |f_n^{(k)}(\psi(t))| \cdot |\psi'(t)|^k \right. \\
&\qquad \left. - \left|\sum_{j=1}^{k-1} b_j f_n^{(j)}(\psi(t)) \psi^{(i_1)}(t) \cdot \cdots \cdot \psi^{(i_j)}(t)\right| : a \le t \le b \right\},
\end{aligned}$$

where (b_j) are the constants given by the chain rule of order k (see for instance [7, I.7.10 Corollary]) and $i_1, \ldots, i_j$ are positive integers with $i_1 + \cdots + i_j = k$.

We can modify h if necessary, so that:

$$\left|\sum_{j=1}^{k-1} b_j h^j \psi^{(i_1)}(t) \cdot \cdots \cdot \psi^{(i_j)}(t)\right| \le \frac{m^k}{2} \qquad (t \in [a, b]).$$

Then

$$\sup\{|(f_n \circ \psi)^{(j)}(t)| : 0 \le j \le k;\ a \le t \le b\} \ge m^k/2$$

and

$$\left\|\sum a_n B f_n\right\| \ge \frac{m^k}{2} \cdot \sup_n |a_n|.$$

Therefore, B preserves a copy of c_0, and hence, it is not weakly compact. Let now $R\colon C^k(\mathbb{R}) \to C^k[a, b]$ be the restriction homomorphism. Then we have $R \circ A = B$, so A is not weakly compact. $\square$

2. **Theorem.** *Given $k \in \mathbb{N}\backslash\{0\}$, let $A\colon C^k(E) \to C^k(F)$ be a nonzero continuous algebra homomorphism when $C^k(E)$, $C^k(F)$ are both endowed with one of the natural topologies, and let $\varphi\colon F \to E$ be its inducing function. Then the following assertions are equivalent*:

(a) *φ is constant*;
(b) *A has one-dimensional rank*;
(c) *A is compact*;
(d) *A is weakly compact.*

Proof. (a) $\Rightarrow$ (b) like in Proposition 1.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d). They are clear.

(d) $\Rightarrow$ (a). Suppose φ is not constant. Then there is $y_0 \in F$ with $\varphi(y_0) \neq \varphi(0)$. Choose $x^* \in E^*$ such that $\langle \varphi(0), x^* \rangle \neq \langle \varphi(y_0), x^* \rangle$, and let $i\colon \mathbb{R} \to F$ be given by $i(\lambda) = \lambda y_0$. If $\overline{\varphi} = x^* \circ \varphi \circ i$, then $\overline{\varphi}$ belongs to $C^k(\mathbb{R})$ and satisfies $\overline{\varphi}(0) \neq \overline{\varphi}(1)$. Let $\overline{A}\colon C^k(\mathbb{R}) \to C^k(\mathbb{R})$ be the homomorphism given by $\overline{A}f = f \circ \overline{\varphi}$. By the previous proposition, $\overline{A}$ is not weakly compact. If $I\colon C^k(\mathbb{R}) \to C^k(E)$ is defined by $Ig = g \circ x^*$ $(g \in C^k(\mathbb{R}))$ and $R\colon C^k(F) \to C^k(\mathbb{R})$ is the restriction homomorphism given by $Rf = f \circ i$, then $\overline{A} = R \circ A \circ I$. Since $\overline{A}$ is not weakly compact, neither is A. □

3. *Remark.* For $k \in \mathbb{N}$, let $C^k_{wu}(E)$ denote the class of all $f \in C^k(E)$ such that f, $d^j f$, and $d^j f(x)$ $(x \in E,\ 1 \leq j \leq k)$ are weakly uniformly continuous on bounded subsets of E. In [6, 11.2.6] it is proved that if E^* has the bounded approximation property, then every nonzero algebra homomorphism $A\colon C^k_{wu}(E) \to C^k_{wu}(F)$ is automatically continuous and induced by a function $\varphi\colon F \to E^{**}$ such that $\phi \circ \varphi \in C^k_{wu}(F)$ for each $\phi \in E^*$. Applying the proof of Theorem 2 to this case, we conclude that A is weakly compact if and only if φ is constant. For $k = 0$, this result is contained in [5].

4. *Remark.* Theorem 2 fails for homomorphisms $A\colon C^k(E) \to C^m(F)$ with $k > m$. Indeed, suppose $\varphi\colon F \to E$ is a nonconstant mapping that factors through a finite-dimensional Banach space G:

$$F \xrightarrow{\xi} G \xrightarrow{\zeta} E \qquad (\varphi = \zeta \circ \xi),$$

where ξ is of class C^m and ζ is of class C^k (k finite or infinite, $k > m$). Then the homomorphism $A\colon C^k(E) \to C^m(F)$ given by $Af = f \circ \varphi$ $(f \in C^k(E))$ factors in the following way:

$$C^k(E) \xrightarrow{A_\zeta} C^k(G) \xrightarrow{i} C^m(G) \xrightarrow{A_\xi} C^m(F),$$

where

$$A_\zeta(f) = f \circ \zeta \qquad (f \in C^k(E)),$$
$$A_\xi(g) = g \circ \xi \qquad (g \in C^m(G)),$$

and i is the identity map, which is known to be compact. So, A is compact too.

An analogous remark can be made for $k = m = \infty$.

However, for $k < m$ every continuous homomorphism from $C^k(E)$ to $C^m(F)$ is induced by a constant mapping $F \to E$ (see [6, 11.2.7]).

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