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REFLEXIVITY OF COMMUTATIVE SUBSPACE LATTICES

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ABSTRACT. A short proof is given of Arveson's reflexivity theorem for strongly closed commutative subspace lattices.

When $\mathscr{L}$ is a lattice of commuting selfadjoint projections on a Hilbert space $\mathscr{H}$, one can form the algebra $\operatorname{Alg}\mathscr{L}$ of all $T \in \mathscr{B}(\mathscr{H})$ for which $TP = PTP$ for all $P \in \mathscr{L}$. Given a subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{H})$ one can form the lattice $\operatorname{Lat}\mathscr{A}$ consisting of all selfadjoint projections P such that $TP = PTP$ whenever $T \in \mathscr{A}$. It is a theorem of W. B. Arveson [1] that if $\mathscr{L}$ is closed in the strong operator topology, then $\mathscr{L}$ is *reflexive*, that is to say that $\mathscr{L} = \operatorname{Lat}\operatorname{Alg}\mathscr{L}$. In this note we shall give a short proof of this result. Our approach avoids topological measure theory and disintegration of measures, though we do use, in a different guise, the class $\mathbf{A}$ of "pseudo-integral" operators that is the key to Arveson's original proof. Other proofs of the theorem have been given by K. R. Davidson [2] and by V. S. Shul'man [3].

Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space. We shall write $\mathbf{A}(\mu)$ for the algebra of all linear operators $T : L^2(\mu) \to L^2(\mu)$ which are bounded from the L^1-norm to the L^1-norm and from the L^∞-norm to the L^∞-norm. For $T \in \mathbf{A}(\mu)$ we define $\|T\|_{\mathbf{A}} = \max\{\|T : L^1 \to L^1\|, \|T : L^\infty \to L^\infty\|\}$. By interpolation, the norm of T in $\mathscr{B}(L^2(\mu))$ is less than or equal to $\|T\|_{\mathbf{A}}$, so that the unit ball of $\|\cdot\|_{\mathbf{A}}$ is a subset of the unit ball of $\mathscr{B}(L^2(\mu))$ for the operator norm. In fact, it a closed subset for the weak operator topology (and hence compact for that topology) since $\|T\|_{\mathbf{A}} \le 1$ if and only if $(Tf \mid g) \le 1$ whenever the elements f, g of $L^2(\mu)$ satisfy $\|f\|_1 \le 1$ and $\|g\|_\infty \le 1$ or $\|f\|_\infty \le 1$ and $\|g\|_1 \le 1$.

When $\mathscr{L}$ is a sublattice of the σ-algebra $\mathscr{F}$ the projections $P_L : f \mapsto f.\mathbf{1}_L$ $(L \in \mathscr{L})$ form a lattice in $\mathscr{B}(L^2(\mu))$. We abuse notation by writing $\operatorname{Alg}\mathscr{L}$ for $\operatorname{Alg}\{P_L : L \in \mathscr{L}\}$.

Theorem. *Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space and let $\mathscr{L}$ be a sublattice of $\mathscr{F}$. For any $F \in \mathscr{F}$*

$$\inf_{L \in \mathscr{L}} \mu(F \triangle L) = \max\{(T\mathbf{1}_F \mid \mathbf{1}_{\Omega\setminus F}) : T \in \mathbf{A}(\mu) \cap \operatorname{Alg}\mathscr{L},\ \|T\|_{\mathbf{A}} = 1\}.$$

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Proof. If $L \in \mathscr{L}$ and $T \in \mathbf{A}(\mu) \cap \operatorname{Alg}\mathscr{L}$, then

$$\begin{aligned}(T\mathbf{1}_F \mid \mathbf{1}_{\Omega\setminus F}) &\leq (T\mathbf{1}_{F\cap L} \mid \mathbf{1}_{\Omega\setminus(L\cup F)}) + (T\mathbf{1}_{F\cap L} \mid \mathbf{1}_{L\setminus F}) + (T\mathbf{1}_{F\setminus L} \mid \mathbf{1}_{\Omega\setminus F}) \\ &\leq 0 + \|T : L^\infty \to L^\infty\|\, \mu(L\setminus F) + \|T : L^1 \to L^1\|\, \mu(F\setminus L) \\ &\leq \|T\|_{\mathbf{A}}\mu(L\triangle F),\end{aligned}$$

so that one inequality ($\geq$) is easily established. To establish the other, we start with the case where $\mathscr{L}$ is a finite lattice.

Lemma. *Let $(\Omega, \mathscr{F}, \mu)$ be a finite measure space and let $\mathscr{L}$ be a finite sublattice of $\mathscr{F}$. For any $F \in \mathscr{F}$ there exist $L \in \mathscr{L}$ and $T \in \operatorname{Alg}\mathscr{L}$ with $\|T\|_{\mathbf{A}} = 1$ such that*

$$\mu(F\triangle L) = (T\mathbf{1}_F \mid \mathbf{1}_{\Omega\setminus F}).$$

Proof. Let $\mathscr{G}$ be the algebra generated by $\mathscr{L}\cup\{F\}$ and let $A_1, \dots, A_m$ be the atoms of $\mathscr{G}$ that are contained in F, $B_1, \dots, B_n$ the atoms that are disjoint from F. Let G be the set of pairs (i, j) such that there is *no* $L \in \mathscr{L}$ with $A_i \subseteq L$, $B_j \cap L = \varnothing$. If $\mathbf{x} = (x_{i,j})_{(i,j)\in G}$ is a family of positive real numbers then we may define an operator $T_{\mathbf{x}}$ in $\operatorname{Alg}\mathscr{L}$ by

$$T_{\mathbf{x}} f = \sum_{(i,j)\in G} \frac{x_{i,j}}{\mu(A_i)\mu(B_j)} (f \mid \mathbf{1}_{A_i})\, \mathbf{1}_{B_j}.$$

We easily calculate the norms

$$\|T_{\mathbf{x}} : L^1 \to L^1\| = \max_i \sum_j \frac{x_{i,j}}{\mu(A_i)},$$

$$\|T_{\mathbf{x}} : L^\infty \to L^\infty\| = \max_j \sum_i \frac{x_{i,j}}{\mu(B_j)}$$

as well as the quantity

$$(T_{\mathbf{x}}\mathbf{1}_F \mid \mathbf{1}_{\Omega\setminus F}) = \sum_{(i,j)\in G} x_{i,j}.$$

Let δ be the maximum of this quantity for $\mathbf{x}$ as above and $\|T_{\mathbf{x}}\|_{\mathbf{A}} \leq 1$. We shall have proved the lemma if we find an element L of $\mathscr{L}$ with $\mu(F\triangle L) \leq \delta$.

Now δ is thus the solution of the following linear programming problem: Maximize $\sum_{(i,j)\in G} x_{i,j}$ subject to $x_{i,j} \geq 0$, $\sum_j x_{i,j} \leq \alpha_i$ for all i, and $\sum_i x_{i,j} \leq \beta_j$ for all j, where $\alpha_i = \mu(A_i)$ and $\beta_j = \mu(B_j)$. This may be regarded as a network-flow problem: we consider a directed graph whose nodes are $A_1, A_2, \dots, A_m$, $B_1, B_2, \dots, B_n$ together with a "source" S and a "sink" T. For each i there is a channel from S to A_i with maximum capacity α_i, for each j there is a channel from B_j to T with capacity β_j, and there is a channel of infinite capacity from A_i to B_j whenever $(i, j) \in G$. Our problem is to find the maximal flow through this network. By the Min-Cut Max-Flow Theorem this maximal flow equals

$$\min_C \sum_{c\in C} \text{capacity of } c,$$

where the minimum is taken over sets C of channels such that S is separated from T if all channels in C are removed from the network. Evidently, we shall not achieve this minimum if we remove a channel of infinite capacity so that the minimizing C will consist of the channels SA_i for i in a certain set I, and the B_jT for j in some J.

We have established the existence of I and J such that

$$\sum_{i\in I}\alpha_i+\sum_{j\in J}\beta_j=\delta$$

and such that for every $(i,j)\in G$ at least one of $i\in I$, $j\in J$ is true. By the definition of G there exists, whenever $i\notin I$ and $j\notin J$, an element $L_{i,j}$ of $\mathscr{L}$ with $A_i\subseteq L_{i,j}$ and $B_j\cap L_{i,j}=\varnothing$. Since $\mathscr{L}$ is a lattice the set

$$L=\bigcap_{j\notin J}\ \bigcup_{i\notin I}L_{i,j}$$

is in $\mathscr{L}$. Also $A_i\subseteq L$ whenever $i\notin I$ and $B_j\cap L=\varnothing$ whenever $j\notin J$, so that

$$\mu(F\triangle L)\le\sum_{i\in I}\mu(A_i)+\sum_{j\in J}\mu(B_j)=\delta.$$

We can now resume the proof of the theorem. Let $\delta=\inf\{\mu(F\triangle L):L\in\mathscr{L}\}$ and let K be the set of all $T\in\mathbf{A}(\mu)$ such that $\|T\|_{\mathbf{A}}\le 1$ and $(T\mathbf{1}_F\mid\mathbf{1}_{\Omega\setminus F})\ge\delta$. This set is compact in the weak operator topology. If $\mathscr{L}'$ is a finite sublattice of $\mathscr{L}$, then

$$K\cap\operatorname{Alg}\mathscr{L}'=\{T\in K:(Tf\mid g)=0\text{ whenever there exists }L\in\mathscr{L}\text{ such that }\operatorname{supp}f\subseteq L\text{ and }\operatorname{supp}g\subseteq\Omega\setminus L\}$$

is closed in the weak operator topology, and is nonempty by the lemma. By compactness we deduce that $K\cap\operatorname{Alg}\mathscr{L}$ is nonempty, which is what we wanted to prove. □

Corollary. *Let $\mathscr{H}$ be a Hilbert space and let $\mathscr{L}$ be a strongly closed lattice of commuting selfadjoint projections on $\mathscr{H}$. Then* $\operatorname{Lat}\operatorname{Alg}\mathscr{L}=\mathscr{L}$.

Proof. The reduction of the problem on a general Hilbert space to the measure-theoretic version that we have just been looking at is rather standard. Let Q be a selfadjoint projection that is not in $\mathscr{L}$; we have to show that Q is not in $\operatorname{Lat}\operatorname{Alg}\mathscr{L}$. Let $\mathscr{M}$ be a maximal abelian selfadjoint subalgebra of $\mathscr{B}(\mathscr{H})$ containing the lattice $\mathscr{L}$; then $\operatorname{Alg}\mathscr{L}\supseteq\mathscr{M}$, so that $\operatorname{Lat}\operatorname{Alg}\mathscr{L}\subseteq\operatorname{Lat}\mathscr{M}$. It is known that $\operatorname{Lat}\mathscr{M}\subset\mathscr{M}$ so that we can certainly assume that $Q\in\mathscr{M}$. Since $\mathscr{L}$ is strongly closed, there exist $f_1,\dots,f_m\in\mathscr{H}$ such that

$$\max_{i\le m}\|(P-Q)f_i\|\ge 1$$

for all $P\in\mathscr{L}$. Let $\mathscr{H}_0$ be the closed subspace generated by $\mathscr{M}\{f_1,\dots,f_m\}$. The orthogonal projection P_0 of $\mathscr{H}$ onto $\mathscr{H}_0$ is in $\mathscr{M}$ and $\mathscr{M}_0=\mathscr{M}|_{\mathscr{H}_0}$ is a maximal abelian selfadjoint subalgebra of $\mathscr{B}(\mathscr{H}_0)$. We can regard $\mathscr{H}_0$ as $L^2(\mu)$ and identify $\mathscr{M}_0$ with $L^\infty(\mu)$ for a suitable finite measure μ. The lattice $\mathscr{L}|_{\mathscr{H}_0}$ of idempotents in $\mathscr{M}_0$ has the form $\{P_L:L\in\mathscr{L}_0\}$ for some sublattice $\mathscr{L}_0$ of $\mathscr{F}$. The restriction to $\mathscr{H}_0$ of Q is P_F for some $F\in\mathscr{F}$. The existence in $\mathscr{H}_0=L^2(\mu)$ of the elements $f_1,\dots,f_m$, implies that there is some $\delta>0$ such that $\mu(F\triangle L)\ge\delta$ for all $L\in\mathscr{L}_0$. The theorem gives some $T_0\in\mathscr{B}(\mathscr{H}_0)$ such that $T_0P_L=P_LT_0P_L$ for all $L\in\mathscr{L}_0$ but $T_0P_F\ne P_FT_0P_F$. If we define $T\in\mathscr{B}(\mathscr{H})$ by $T=T_0P_0$, then $T\in\operatorname{Alg}\mathscr{L}$ but $TQ\ne QTQ$. □

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