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DENSITY OF THE POLYNOMIALS IN THE HARDY SPACE OF CERTAIN SLIT DOMAINS

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ABSTRACT. In this article we construct a Jordan arc Γ in the complex plane, with endpoints 0 and 1, such that the polynomials are dense in the Hardy space $H^2(\mathbb{D}\backslash\Gamma)$; $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

It is well known that if $G = \{z \in \mathbb{C} : |z| < 1\}\backslash[0, 1]$ ($\mathbb{C}$ denotes the complex plane), then the polynomials are not dense in the Hardy space $H^2(G)$. One of the assertions of this paper, however, is that there are regions D of the same sort as G such that the polynomials are dense in $H^2(D)$. In fact, we construct a homeomorphic image Γ of the interval $[0, 1]$, where Γ has endpoints 0 and 1, and $\Gamma\backslash\{1\} \subseteq \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, such that the polynomials are dense in $H^2(\mathbb{D}\backslash\Gamma)$.

Recall that if D is a bounded Dirichlet region, then the Hardy space $H^2(D)$ is the collection of functions f that are analytic in D such that $|f|^2$ has a harmonic majorant on D . Furthermore, for any point z_0 in D (norming point), the mapping $\|\cdot\|_{z_0}: H^2(D) \to \mathbb{R}$ defined by $\|f\|_{z_0} = (u_f(z_0))^{1/2}$, where u_f is the least harmonic majorant of $|f|^2$ on D , is a norm on $H^2(D)$, and, under this norm, $H^2(D)$ forms a Banach space (cf. [6]). By Harnack's inequality, different norming points yield equivalent norms. We let $\omega(\cdot, D, z_0)$ denote *harmonic measure* on ∂D evaluated at z_0 . Notice that if f is analytic on D and continuous on $\overline{D}$, then $f \in H^2(D)$ and

$$\|f\|_{z_0} = \left\{\int |f(\zeta)|^2 \, d\omega(\zeta, D, z_0)\right\}^{1/2} .$$

1. **Definition.** A function $\gamma: [0, 1] \to \mathbb{C}$ is said to be a *Jordan arc* if and only if it is both continuous and one-to-one. Throughout this paper we shall identify a Jordan arc γ with its *trace* $\Gamma := \gamma([0, 1])$.

In order to minimize technical details we do much of our work on a particular "annular" region which has rectilinear boundary. For the rest of the paper let $E = \{z = x+iy : 1 < \max\{|x|, |y|\} < 2\}$, $S = \{z = x+iy : \max\{|x|, |y|\} = 1\}$, and $T = \{z = x + iy : \max\{|x|, |y|\} = 2\}$. Let us say that a Jordan arc $\Gamma :=$

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$\gamma([0, 1])$ *connects* S to T if $\gamma(0) \in S$, $\gamma(t) \in E$ for $0 < t < 1$, and $\gamma(1) \in T$. If z and ζ are complex numbers, then let $[z, \zeta] = \{(1-t)z + t\zeta : 0 \leq t \leq 1\}$ (observe that $[z, \zeta] = [\zeta, z]$) be the *segment* that connects z to ζ.

2. **Definition.** Let G be a bounded, simply connected region in $\mathbb{C}$. A Jordan arc γ is called a *cross-cut* of G if both $\gamma(0)$ and $\gamma(1)$ are in ∂G and $\gamma((0, 1)) \subseteq G$.

3. **Lemma.** (a) *Let* D *and* G *be bounded, simply connected regions in* $\mathbb{C}$ *such that* $z_0 \in D \subseteq G$. *If* B *is a Borel subset of* $(\partial D) \cap (\partial G)$, *then* $\omega(B, D, z_0) \leq \omega(B, G, z_0)$.

(b) *Let* G *be a bounded, simply connected region in* $\mathbb{C}$. *If* γ *is a cross-cut of* G $(\Gamma = \gamma([0, 1]))$, *the components of* $G \backslash \Gamma$ *are* G_1 *and* G_2, $z_0 \in G_1$ *and* $F = (\partial G_2) \backslash \Gamma$, *then* $\omega(F, G, z_0) \leq \omega(\Gamma, G_1, z_0)$.

Proof (sketch). Part (a) follows from the maximum principle for harmonic functions. Part (b) is a consequence of (a) and the fact that harmonic measure is a probability measure.

4. **Lemma.** *Suppose* $0 < \varepsilon < 1/4$ *and* $[\xi, \eta]$ *is a segment of length* 2ε *in* $\{z \in E : \mathrm{Re}(z) > 0\}$. *Then* $\omega([\xi, \eta], E \backslash [\xi, \eta], -3/2) \leq -1/\log(\varepsilon)$.

Proof. Let $D = \{z \in \mathbb{C} : |z| < 4\}$ and $\Delta = \{z \in \mathbb{C} : |z - ((\zeta + \eta)/2)| < \varepsilon\}$; notice that $\overline{\Delta} \subseteq \{z \in \mathbb{C} : |z| < 3\}$. Since $E \backslash [\zeta, \eta] \subseteq D \backslash [\zeta, \eta]$, it follows from Lemma 3(a) that $\omega([\zeta, \eta], E \backslash [\zeta, \eta], -3/2) \leq \omega([\zeta, \eta], D \backslash [\zeta, \eta], -3/2)$. Likewise, since $D \backslash \overline{\Delta} \subseteq D \backslash [\zeta, \eta]$ we have $\omega(\partial D, D \backslash \overline{\Delta}, -3/2) \leq \omega(\partial D, D \backslash [\zeta, \eta], -3/2)$, and therefore $\omega([\zeta, \eta], D \backslash [\zeta, \eta], -3/2) \leq \omega(\partial \Delta, D \backslash \overline{\Delta}, -3/2)$. Consequently, $\omega([\zeta, \eta], E \backslash [\zeta, \eta], -3/2) \leq \omega(\partial \Delta, D \backslash \overline{\Delta}, -3/2)$.

Next we let φ be a Möbius transformation that maps D onto the unit disk $\mathbb{D}$ and Δ onto a disk Δ_φ with center $z = 0$. Elementary calculations give us that $|\varphi(-3/2)| \geq 3/8$ and that the radius of Δ_φ is at most ε. So

$$\begin{aligned}\log(3/8) \leq \log|\varphi(-3/2)| &= \int \log|z| \, d\omega(z, \mathbb{D} \backslash \overline{\Delta}_\varphi, \varphi(-3/2)) \\ &= [\log(\text{radius } (\Delta_\varphi))] \cdot \omega(\partial \Delta_\varphi, \mathbb{D} \backslash \overline{\Delta}_\varphi, \varphi(-3/2)) \\ &\leq [\log(\varepsilon)] \cdot \omega(\partial \Delta_\varphi, \mathbb{D} \backslash \overline{\Delta}_\varphi, \varphi(-3/2)).\end{aligned}$$

Therefore,

$$\begin{aligned}\omega([\zeta, \eta], E \backslash [\zeta, \eta], -3/2) &\leq \omega(\partial \Delta, D \backslash \overline{\Delta}, -3/2) \\ &= \omega(\partial \Delta_\varphi, \mathbb{D} \backslash \overline{\Delta}_\varphi, \varphi(-3/2)) \leq \frac{\log(3/8)}{\log(\varepsilon)} < \frac{-1}{\log(\varepsilon)}. \quad \square\end{aligned}$$

5. **Lemma.** *If* Γ *is a Jordan arc that connects* S *to* T, $\omega := \omega(\cdot, E \backslash \Gamma, z_0)$, *and* $1/z$ *can be approximated by polynomials in the* $L^2(\omega)$ *norm, then the polynomials are dense in the Hardy space* $H^2(E \backslash \Gamma)$.

Proof. Let φ be a conformal map from $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ one-to-one and onto $E \backslash \Gamma$ such that $\varphi(0) = z_0$, and define $\|\cdot\| : H^2(E \backslash \Gamma) \to \mathbb{R}$ by

$$\|f\|^2 = |f(z_0)|^2 + \int_{E \backslash \Gamma} |f'|^2 (1 - |\varphi^{-1}|^2) \, dA.$$

By Green's Theorem and a change of variables, $\|\cdot\|$ defines a norm on $H^2(E\backslash\Gamma)$ that is equivalent to the Hardy space $H^2(E\backslash\Gamma)$ norm.

Now from our hypothesis it follows that no point in $\partial(E\backslash\Gamma)$ can be an analytic bounded point evaluation for the polynomials with respect to the $L^2(\omega)$ norm. Since the $L^2(\omega)$ and $H^2(E\backslash\Gamma)$ norms are equivalent for the polynomials, we can conclude that no point in $\partial(E\backslash\Gamma)$ is an analytic bounded point evaluation for the polynomials with respect to the $L^2((1-|\varphi^{-1}|^2)\,dA)$ norm. Therefore, by [5, Theorem 4], the polynomials are dense in $L^2_a(E\backslash\Gamma, (1-|\varphi^{-1}|^2)\,dA)$, and thus are dense in $H^2(E\backslash\Gamma)$ by [4, Corollary 3.4]. □

Let Γ be a Jordan arc that connects S to T. How pathological must Γ be so that the polynomials have a chance of being dense in $H^2(E\backslash\Gamma)$? If φ is a conformal map from the unit disk $\mathbb{D}$ one-to-one and onto $E\backslash\Gamma$ and the polynomials are dense in $H^2(E\backslash\Gamma)$, then by [4, Corollary 3.5] φ must be univalent almost everywhere on $\partial\mathbb{D}$. This can be rephrased in terms of $\omega(\cdot, E\backslash\Gamma, z_0)$ to give us that the set of tangent points of Γ (see [3]) has one-dimensional Hausdorff measure equal to zero; but much more can be said. Indeed, if there exists one point z in Γ and a crescent Ω in $E\backslash\Gamma$, with multiple boundary point z, such that the bounded component of $\mathbb{C}\backslash\overline{\Omega}$ contains $\{z = x+iy : \min\{|x|, |y|\} \le 1\}$ and the polynomials are not dense ir $H^2(\Omega)$, then the polynomials are not dense in $H^2(E\backslash\Gamma)$. A consequence of this (cf. [1]) is that if there exists one point z in Γ such that from each side of Γ we can approach z through a cone in $E\backslash\Gamma$, then the polynomials are not dense in $H^2(E\backslash\Gamma)$.

6. **Theorem.** *There exists a Jordan arc Γ that connects S to T such that the polynomials are dense in $H^2(E\backslash\Gamma)$.*

Proof. By Lemma 5, it is sufficient to produce a Jordan arc Γ that connects S to T such that $-3/2 \notin \Gamma$ and $1/z$ can be approximated by polynomials in the $L^2(\omega)$ norm; $\omega := \omega(\cdot, E\backslash\Gamma, -3/2)$. A reasonable strategy for producing this Γ is to find a sequence of polynomials $\{p_n\}$ and a sequence of polygonal Jordan arcs $\{\Gamma_n\}$ such that

(6.1)
- (a) for all n, Γ_n connects S to T, $\mathrm{Re}(z) > 0$ for all z in Γ_n, and $\{\Gamma_n\}$ converges uniformly to a Jordan arc Γ that connects S to T;
- (b) $\displaystyle\int |1/z - p_k|^2\,d\omega_n < 1/k$ whenever $1 \le k \le n$, where $\omega_n := \omega_n(\cdot, E\backslash\Gamma_n, -3/2)$.

In fact, for convenience of proof, we shall choose Γ_n so that its angle of incidence with both S and T is $\pi/2$ and that the angle formed by Γ_n at any of its vertices is at least $\pi/3$. The limiting arc Γ is the one we are after.

Let $W_1 = \{z \in E : \mathrm{dist}(z, [1, 2]) < 1/8\}$. By Runge's Theorem, there is a polynomial p_1 such that

$$\|(1/z - p_1)^2\|_{E\backslash W_1} := \sup\{|1/z - p_1(z)|^2 : z \in E\backslash W_1\} < 1/2.$$

Now we construct Γ_1. Let $\Omega_1 = \{1 - i/4, 5/4 + i/4, 3/2 - i/4, 7/4 + i/4, 2 - i/4\}$; obviously $5 =$ cardinality of $\Omega_1 := |\Omega_1|$. Choose $0 < \varepsilon_1 < 1/16$ small

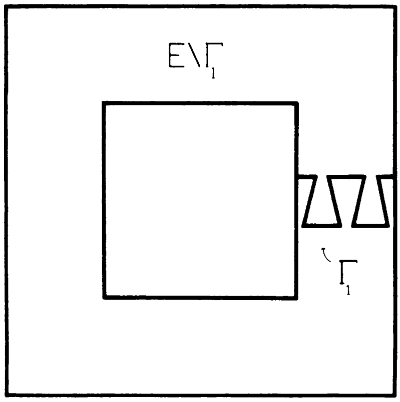

FIGURE 1

enough so that

$$(6.2) \qquad 5\left(\frac{-1}{\log(\varepsilon_1)}\right) < 1/(2\|(1/z-p_1)^2\|_{W_1}).$$

Let $K_1 = ([1+i/4, 2+i/4]\cup[1-i/4, 2-i/4])\setminus\bigcup_{z\in\Omega_1} B(z;\varepsilon_1)$, where $B(z;\varepsilon_1) := \{\zeta \in \mathbb{C} : |z-\zeta| < \varepsilon_1\}$. Now K_1 is a closed set with five components: $I_1(1)$, $I_1(3)$, $I_1(5)$, $I_1(7)$, and $I_1(9)$, each of which is a segment. The numbering scheme is as follows: $I_1(j) \subseteq [1+i/4, 2+i/4]$ for $j = 1, 5, 9$; $I_1(j) \subseteq [1-i/4, 2-i/4]$ for $j = 3, 7$; $\mathrm{Re}(z) < \mathrm{Re}(\zeta) < \mathrm{Re}(\eta)$ whenever $z \in I_1(1)$, $\zeta \in I_1(5)$, and $\eta \in I_1(9)$, and $\mathrm{Re}(z) < \mathrm{Re}(\zeta)$ whenever $z \in I_1(3)$ and $\zeta \in I_1(7)$. Connect the right endpoint of $I_1(1)$ to the left endpoint of $I_1(3)$, the right endpoint of $I_1(3)$ to the left endpoint of $I_1(5)$, the right endpoint of $I_1(5)$ to the left endpoint of $I_1(7)$, and the right endpoint of $I_1(7)$ to the left endpoint or $I_1(9)$, with segments $I_1(2)$, $I_1(4)$, $I_1(6)$, and $I_1(8)$, respectively. Let $\Gamma_1 = \bigcup_{j=1}^{9} I_1(j)$ (see Figure 1). Notice that Γ_1 is a polygonal Jordan arc that connects S to T, the angle of incidence of Γ_1 with both S and T is $\pi/2$, and the angle formed by Γ_1 at any of its vertices is at least $\pi/3$.

Now $\overline{W}_1$ is only accessible in $E\setminus\Gamma_1$ from $z = -3/2$ through five "gaps" in Γ_1, each of size at most $2\varepsilon_1$. Consequently, by (6.2), Lemma 4, and Lemma 3(b), $\omega_1(\overline{W}_1) < 1/(2\|(1/z-p_1)^2\|_{W_1})$; $\omega_1 := \omega_1(\cdot, E\setminus\Gamma_1, -3/2)$. Therefore, since $\|(1/z-p_1)^2\|_{E\setminus W_1} < 1/2$, we have that

$$\int |1/z - p_1|^2\, d\omega_1 < 1.$$

For $n \geq 2$, Γ_n is constructed inductively so that "over" each segment of Γ_{n-1}, Γ_n looks like Γ_1. In order to construct Γ_n, certain other items need to be defined inductively. For $n \geq 2$, let

$$W_n = \{z \in E : \mathrm{dist}(z, \Gamma_{n-1}) < \varepsilon_{n-1}/16\}$$

and let

$$W_n' = \{z \in E : \mathrm{dist}(z, \Gamma_{n-1}) < \varepsilon_{n-1}/8\}.$$

By Runge's Theorem there is a polynomial p_n such that $\|(1/z - p_n)^2\|_{E\setminus W_n} < 1/(2n)$. The substance of the inductive step is found in the construction of Γ_2 and so, for the most part, we focus our attention there.

Recall that $\Gamma_1 = \bigcup_{j=1}^{9} I_1(j)$. For $1 \le j \le 9$, let $I_1^*(j)$ be the straight line that contains $I_1(j)$, and let $D_2(j) = \{z \in \mathbb{C} : \operatorname{dist}(z, I_1^*(j)) < \varepsilon_1/8\}$. For $j = 2, 4, 6, 8$, let $V_2(j, j+1) = D_2(J) \cap D_2(j+1)$ and $V_2(j, j-1) = D_2(j) \cap D_2(j-1)$. Let

$$W_2'' = W_2' \cup \left\{ \bigcup_{j=1}^{4} (V_2(2j, 2j+1) \cup V_2(2j, 2j-1)) \right\}.$$

Notice that, unlike $\partial W_2'$, $\partial W_2''$ is a polygon. Moreover, since the angle formed by Γ_1 at any of its vertices is at least $\pi/3$, it follows that $\operatorname{dist}(z, \Gamma_1) < \varepsilon_1/4$ whenever $z \in W_2''$. We shall construct Γ_2 using $\operatorname{cl}\{E \cap (\partial W_2'')\}$.

Now $\partial V_2(j, j+1)$ [resp., $\partial V_2(j, j-1)$] is a parallelogram ($j = 2, 4, 6, 8$). Let $a_2(j, j+1)$ [resp., $a_2(j, j-1)$] be the unique vertex of $\partial V_2(j, j+1)$ [resp., $\partial V_2(j, j-1)$] which is in $co(I_1(j) \cup I_1(j+1)) :=$ closed convex hull of $(I_1(j) \cup I_1(j+1))$ [resp., $co(I_1(j) \cup I_1(j-1))$]. Let $b_2(j, j+1)$ [resp., $b_2(j, j-1)$] be the unique point in $\partial D_2(j+1)$ [resp., $\partial D_2(j-1)$] such that the segment $[a_2(j, j+1), b_2(j, j+1)]$ [resp., $[a_2(j, j-1), b_2(j, j-1)]$] is perpendicular to $I_1^*(j+1)$ [resp., $I_1^*(j-1)$], and let $c_2(j, j+1)$ [resp., $c_2(j, j-1)$] be the unique point in $\partial D_2(j)$ such that the segment $[a_2(j, j+1), c_2(j, j+1)]$ [resp., $[a_2(j, j-1), c_2(j, j-1)]$] is perpendicular to $I_1^*(j)$. Let $a_2(0, 1)$ [resp., $a_2(10, 9)$] be the intersection of the component of $\operatorname{cl}\{E \cap (\partial W_2'')\}$ that contains $a_2(2, 1)$ with S [resp., T], and let $b_2(0, 1)$ [resp., $b_2(10, 9)$] be the intersection of the component of $\operatorname{cl}\{E \cap (\partial W_2'')\}$ that contains $b_2(2, 1)$ with S [resp., T]. For $1 \le j \le 9$, if j is odd, then let $R_2(j)$ be the rectangle with vertices $a_2(j+1, j)$, $a_2(j-1, j)$, $b_2(j+1, j)$, and $b_2(j-1, j)$; and if j is even, then let $R_2(j)$ be the rectangle with vertices $a_2(j, j+1)$, $a_2(j, j-1)$, $c_2(j, j+1)$, and $c_2(j, j-1)$. Call a rectangle $R_2(j)$ *even* if its a_2-vertices are diagonal, and *odd* otherwise. Notice that $R_2(j)$ is even if j is odd, and odd if j is even.

With straight lines that are perpendicular to $I_2^*(j)$, partition $R_2(j)$ into congruent subrectangles so that the number of subrectangles is even [resp., odd] if $R_2(j)$ is even [resp., odd], the greatest dimension of any subrectangle is $\varepsilon_1/4$ (the width of $R_2(j)$) and the least dimension is no less than $\varepsilon_1/8$; it is possible to partition in this way because the length of any $R_2(j)$ is at least twice its width—see Figure 2 on next page (labeled in part). Let Ω_2 be the collection of points defined by:

(i) $a_2(0, 1) \in \Omega_2$

(ii) $z \in \Omega_2$ if and only if z is a vertex of some subrectangle of some $R_2(j)$ and the vertex diagonal to z in this subrectangle is in Ω_2.

Now choose $0 < \varepsilon_2 < \varepsilon_1/16$ (for $n \ge 2$, $0 < \varepsilon_n < \varepsilon_{n-1}/16$) small enough so that

$$(6.3) \qquad |\Omega_2|(-1/\log(\varepsilon_2)) < 1/(4\|(1/z - p_2)^2\|_{W_2}),$$

and let $K_2 = \operatorname{cl}\{E \cap (\partial W_2'')\} \setminus \bigcup_{z \in \Omega_2} B(z; \varepsilon_2)$. Notice that K_2 is made up of finitely many components, each of which is either a segment or a polygonal Jordan arc that is the union of two segments. In the same way that Γ_1 was pieced

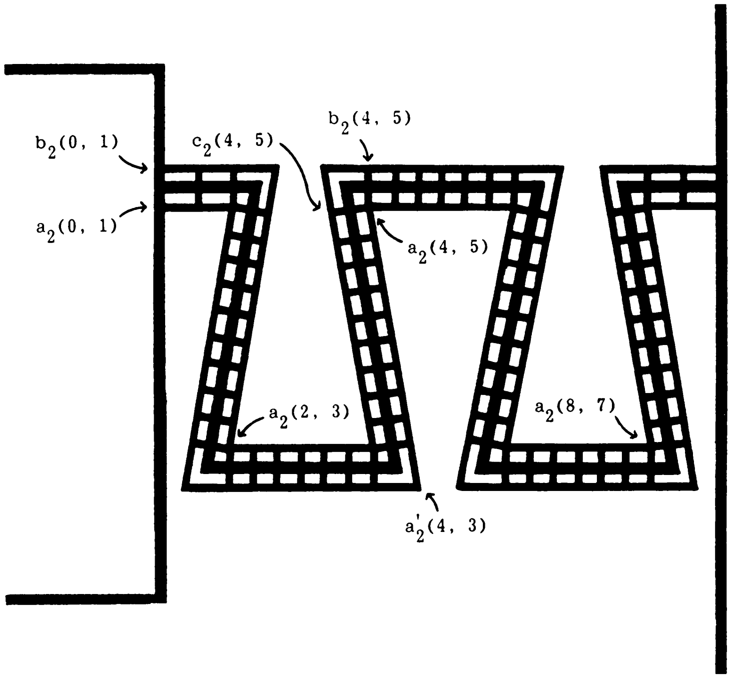

FIGURE 2

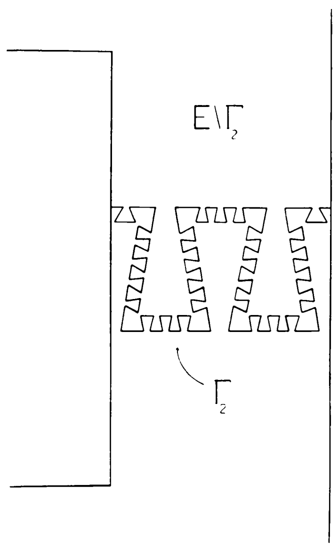

FIGURE 3

together, construct Γ_2 by connecting, with segments, the right endpoint of the component of K_2 which contains $b_2(0, 1)$ to the left endpoint of the component which contains the vertex that is diagonal to $b_2(0, 1)$ in the subrectangle of $R_2(1)$ which has $b_2(0, 1)$ as a vertex, etc. (see Figure 3). The resulting arc Γ_2 is a polygonal Jordan arc whose angle of incidence with both S and T is

$\pi/2$ and whose angle at any vertex is at least $\pi/3$. Moreover, any maximal segment of Γ_2 (i.e., a segment of Γ_2 that is properly contained in no other segment of Γ_2) has length at least $2\varepsilon_2$. Now since $\|(1/z - p_2)^2\|_{E\backslash W_2} < 1/4$ and $\overline{W}_2$ is only accessible in $E\backslash\Gamma_2$ from $z = -3/2$ through $|\Omega_2|$ "gaps" in Γ_2 each of size at most $2\varepsilon_2$, it follows from (6.3), Lemma 4, and Lemma 3(b) that

$$\int |1/z - p_2|^2 \, d\omega_2 < 1/2,$$

where $\omega_2 := \omega_2(\cdot, E\backslash\Gamma_2, -3/2)$. Also notice that by our choice of Γ_2, in order to access $\overline{W}_1$ in $E\backslash\Gamma_2$ from $z = -3/2$, one must pass through one of five gaps in Γ_2, each of which represents a narrowing of one of the gaps in Γ_1. Consequently,

$$\int |1/z - p_1|^2 \, d\omega_2 < 1.$$

For $n \geq 3$, p_n is chosen and Γ_n is constructed in basically the same way we chose p_2 and constructed Γ_2.

Let us parametrize Γ_n. Define $\gamma_1 \colon [0, 1] \to \Gamma_1$ by $\gamma_1(x)$ is the point on Γ_1 whose distance along Γ_1 from S is $x \cdot [\text{length}\,(\Gamma_1)]$. Now we turn to Γ_2. For $j = 2, 4, 6, 8$ let $a_2'(j, j+1)$ [resp., $a_2'(j, j-1)$] be the vertex of $\partial V(j, j+1)$ [resp., $\partial V(j, j-1)$] that is diagonal to $a_2(j, j+1)$ [resp., $a_2(j, j-1)$]. Notice that $a_2'(j, j+1)$ and $a_2'(j, j-1)$ are in Γ_2. Now $\Gamma_2\backslash\{\bigcup_{i=1}^4 \{a_2'(2i, 2i+1), a_2'(2i, 2i-1)\}\}$ has nine components; number them as to the order in which each is encountered when traversing Γ_2 from S to T. Define a continuous one-to-one function $\beta_2 \colon \Gamma_1 \to \Gamma_2$ by mapping $I_1(j)$ (recall that $\Gamma_1 = \bigcup_{j=1}^9 I_1(j)$) onto the closure of the jth component of $\Gamma_2\backslash\{\bigcup_{i=1}^4 \{a_2'(2i, 2i+1), a_2'(2i, 2i-1)\}\}$ in the same way that γ_1 maps $[0, 1]$ onto Γ_1. Let $\gamma_2 = \beta_2 \circ \gamma_1$. Similarly, for any n, define a continuous one-to-one function $\beta_n \colon \Gamma_{n-1} \to \Gamma_n$ by mapping any maximal segment of Γ_{n-1} (i.e., a segment of Γ_{n-1} that is properly contained in no other segment of Γ_{n-1}) to the part of Γ_n that "covers" the segment. Then let $\gamma_n = \beta_n \circ \gamma_{n-1}$.

Choose $\delta > 0$. Now there exists $N \geq 3$ such that $\varepsilon_{n-2} < \delta$. No maximal segment of Γ_{n-1} has length greater than $(3/4) \cdot \varepsilon_{N-2}$. So, by the construction of Γ_k and the definition of γ_k, if $m, n \geq N$ and $t \in [0, 1]$, then $|\gamma_m(t) - \gamma_n(t)| < \delta$. Therefore, $\{\gamma_n\}$ is uniformly Cauchy and hence converges uniformly to a continuous function $\gamma \colon [0, 1] \to \Gamma := \gamma([0, 1])$.

To show that γ is one-to-one, choose s and t in $[0, 1]$ such that $s \neq t$. By the definition of γ_n there exists N such that $\gamma_N(s)$ and $\gamma_N(t)$ are in nonadjacent maximal segments of Γ_N. Reviewing the construction of Γ_N, we find that $|\gamma_N(s) - \gamma_N(t)| \geq 2\varepsilon_N$. In fact, if $n \geq N+1$, then

$$|\gamma_n(s) - \gamma_n(t)| \geq 2\varepsilon_N - 2 \cdot \sum_{k=N+1}^{n} \frac{\varepsilon_n}{4^{k-N}} > \varepsilon_N .$$

Hence, $|\gamma_n(s) - \gamma_n(t)| \not\to 0$ as $n \to \infty$, and so $\gamma(s) \neq \gamma(t)$. Therefore γ is one-to-one, and Γ is a Jordan arc.

We now have a sequence of polynomials $\{p_n\}$ and a sequence of polygonal Jordan arcs $\{\Gamma_n\}$ which satisfy (6.1)(a) and (b). Let $\omega := \omega(\cdot, E\backslash\Gamma, -3/2)$.

Since Γ_n converges uniformly to Γ, it follows that, for fixed k,

$$\int |1/z - p_k|^2\, d\omega_n \to \int |1/z - p_k|^2\, d\omega,$$

as $n \to \infty$; $\omega_n := \omega_n(\cdot, E\backslash\Gamma_n, -3/2)$. Therefore, because $\int |1/z - p_k|^2\, d\omega_n < 1/k$ whenever $1 \le k \le n$, we have that

$$\int |1/z - p_k|^2\, d\omega \le 1/k \to 0,$$

as $k \to \infty$. by Lemma 5, the proof is now complete. □

7. **Theorem.** *There exists a Jordan* $\Gamma := \gamma([0, 1])$, *where* $\gamma(0) = 0$, $\gamma(t) \in \mathbb{D} := \{z \in \mathbb{C}: |z| < 1\}$ *for* $0 < t < 1$, *and* $\gamma(1) = 1$, *such that the polynomials are dense in* $H^2(\mathbb{D}\backslash\Gamma)$.

Proof (sketch). In a way similar to the proof of Theorem 4, we produce a sequence of Jordan arcs $\{\Gamma_n := \gamma_n([0, 1])\}$ (Γ_n having the same geometry as in the proof of Theorem 4), where $\gamma_n([0, 1]) \subseteq \{z \in \mathbb{D}: \mathrm{Re}(z) > 0\}$ and $|\gamma_n(1)| = 1$ for all n, a sequence of points $\{t_n\}$, where $0 < t_n < 1$ and $|\gamma_n(t_n) - \gamma_n(0)| \to 0$ as $n \to \infty$, and a sequence of polynomials $\{p_n\}$ such that:

(a) Γ_n converges uniformly to a Jordan arc $\Gamma := \gamma([0, 1])$, where $\gamma(0) = 0$, $\gamma(t) \in \mathbb{D}$ for $0 < t < 1$, and $\gamma(1) = 1$;

(b) $|p_n(\gamma_n(t_n))| \ge 1$ for all n, and $\int |p_k|^2\, d\omega_n < 1/k$ whenever $1 \le k \le n$, where $\omega_n := \omega_n(\cdot, \mathbb{D}\backslash\Gamma_n, -1/2)$.

The limiting arc $\Gamma := \gamma([0, 1])$ will then have the property that, for all n, $\int |p_n|^2\, d\omega \le 1/n$ ($\omega := \omega(\cdot, \mathbb{D}\backslash\Gamma, -1/2)$) and yet $|p_n(\gamma_n(t_n))| \ge 1$, where $\gamma_n(t_n) \to \gamma(0)$ as $n \to \infty$. So, $\gamma(0)$ is not an analytic bounded point evaluation for the polynomials with respect to the $H^2(\mathbb{D}\backslash\Gamma)$ norm, and hence nor is any point in $\partial(\mathbb{D}\backslash\Gamma)$. Following an argument similar to the proof of Lemma 3, we get that the polynomials are dense in $H^2(\mathbb{D}\backslash\Gamma)$. □

8. *Remark.* Theorems 4 and 5 provide us with new examples of analytic Toeplitz operators T_φ, where φ is a Riemann map from the unit disk $\mathbb{D}$ onto $E\backslash\Gamma$ (of Theorem 4) or onto $\mathbb{D}\backslash\Gamma$ (of Theorem 5), such that T_φ is cyclic (with cyclic vector 1) and yet φ is not a weak-star generator of H^∞ (cf. [8]).

There is unfinished business here, and yet very little of it is easily approachable.

9. **Problem.** Find a condition on Γ which is both necessary and sufficient for density of the polynomials in $H^2(\mathbb{D}\backslash\Gamma)$, where $\Gamma := \gamma([0, 1])$ is a Jordan arc such that $\gamma([0, 1)) \subseteq \mathbb{D}$ and $\gamma(1) = 1$.

10. **Question.** Does there exist a Jordan arc Γ, with endpoints 0 and 1, such that the polynomials are dense in $L_a^2(\mathbb{D}\backslash\Gamma, dA)$?

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