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IMBEDDING OF ANY VECTOR FIELD IN A SCALAR SEMILINEAR PARABOLIC EQUATION

P. POLÁČIK

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ABSTRACT. The scalar semilinear parabolic equation

$$u_t = \Delta u + f(x, u, \nabla u), \qquad x \in \Omega, \ t > 0,$$

on a smooth bounded convex domain $\Omega \subset \mathbb{R}^N$ under Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{2}$$

is considered.

For any prescribed vector field H on $\mathbb{R}^N$, a function f is found such that the flow of (1), (2) has an invariant N-dimensional subspace and the vector field generating the flow of (1), (2) on this invariant subspace coincides, in appropriate coordinates, with H.

INTRODUCTION

Consider the boundary-value problem

$$u_t = \Delta u + f(x, u, \nabla u), \qquad x \in \Omega, \ t > 0, \tag{1}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, n is the unit normal vector field on $\partial\Omega$ pointing out of Ω, and $f: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is of class C^1. This problem defines a local dynamical system on the Sobolev space

$$X := W_b^{2,p} = \{u \in W^{2,p} : u \text{ satisfies (2)}\}, \qquad p > N$$

(see [He1, Am]). Though this is an infinite-dimensional dynamical system, it is generated by an equation with special structure, and it is not immediately clear whether some sort of complicated dynamical behavior can be encountered in it. In one dimension $(N = 1)$, for example, bounded trajectories of equations of this type are known to exhibit a very simple behavior, as t approaches infinity. One finds only convergence to equilibria [Ze, Ma] or a kind of Poincaré-Bendixson theorem, if nonseparated boundary conditions are admitted [F-M].

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In this sense, the dynamics of such equations is no more complicated than the dynamics of planar systems. Another result (the Morse-Smale property), which significantly reduces the variety of dynamical systems defined by the one-dimensional problems, has been proved in [An, He3].

On the other hand, in [F-P] a simple nonlocal equation (more specifically, an equation involving a spatial integral of the unknown function) in one space dimension has been shown for which the dynamics can be rated as complicated. In [Po2], our attempt was to show that one can give a similar rate to local equations (1) in higher space dimension. We have proved in that paper, that any finite jet of an arbitrary N-dimensional vector field can be realized in an equation (1) on a ball $\Omega \subset \mathbb{R}^N$, under Dirichlet boundary condition

$$u_{|\partial\Omega} = 0. \tag{3}$$

In other words, for any integer $k > 0$ and any C^k vector field H on $\mathbb{R}^N$ with $H(0) = 0$, one can find a function f such that (1), (3) has an invariant N-dimensional manifold through the equilibrium $u \equiv 0$, and the Taylor expansion of the vector field given by the flow of (1), (3) on this invariant manifold coincides, in appropriate coordinates, with the Taylor expansion of H, up to kth order terms. A consequence of this result is that a hyperbolic invariant $(N-1)$-torus can be found in (1), (3). One would also expect some sort of chaos to occur in (1), (3).

The method of [Po2] can be easily modified to obtain the same result for Neumann problem (1), (2). However, for this problem a significantly stronger property can be established using a simpler procedure. In this paper we prove that any N-dimensional vector field (not just its finite jets) can be imbedded in (1), (2). This result show that "chaos is present" in the equations of the given class. If $N \geq 3$, we can e.g., imbed in (1), (2) a horseshoe of Šilnikov's example (see [Si, G-H]) or a system with the Lorenz attractor (see [G-H]). Let us point out, however, that trajectories with more interesting behavior are in a sense exceptional (and hence can be hardly "observed"). Due to monotonicity of the semiflow of (1), (2), most trajectories converge to an equilibrium [Po1].

The proof we present here is rather elementary (we do not work with perturbations of the center manifold as in [Po2]) and constructive (we give an explicit expression for the equation (1) in terms of the prescribed vector field). Moreover the result is valid for any convex (or at least starshaped) domain.

2. Statement of the result and proof

Recall that Ω is starshaped if there is a point q such that the segment joining $x \in \Omega$ and q lies entirely in Ω. In the sequel we assume that Ω is starshaped with the origin $q = 0$ having the above property. Note that this implies that for any $x \in \partial\Omega$ the vector $x = x - 0$ satisfies

$$\langle x, \nu(x)\rangle := x_1\nu_1(x) + \cdots + x_N\nu_N(x) \geq 0. \tag{4}$$

A subspace $W \subset X$ is said to be invariant for (1), (2) if for any solution $u(t, x)$ of (1), (2) with the initial condition $u(0, \cdot) \in W$ one has $u(t, \cdot) \in W$ for any t in the interval of existence of $u(t, \cdot)$. If $W = \operatorname{span}\{\phi_1, \dots, \phi_N\}$ is an N-dimensional invariant subspace and $H \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, we say that the flow of (1), (2) is generated by H, if for any trajectory $c(t) = (c_1(t), \dots, c_N(t))$ of the equation

$$\dot{c} = H(c)$$

the function

$$u(t, x) = c_1(t)\phi_1(x) + \cdots + c_N(t)\phi_N(x)$$

is a solution of (1), (2).

Theorem. *There exist functions $\phi_1(\cdot), \dots, \phi_N(\cdot) \in X$ with the property that for any $H \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ there is an $f \in C^1(\overline{\Omega} \times \mathbb{R}^N)$ such that the subspace $W := \operatorname{span}\{\phi_1, \dots, \phi_N\}$ is invariant for* (1), (2) *and the flow of* (1), (2) *on W is generated by H.*

The proof, which we give after some preliminary remarks, uses the following simple observation: The space $W = \operatorname{span}\{\phi_1, \dots, \phi_N\}$ is invariant for (1), (2) and the flow of (1), (2) on W is generated by $H = (h_1, \dots, h_N)$ if and only if

$$\sum_{i=1}^{N}(h_i(c)\phi_i(x) - c_i\Delta\phi_i(x)) = f\left(x, \sum_{i=1}^{N} c_i\phi_i(x), \sum_{i=1}^{N} c_i\nabla\phi_i(x)\right) \tag{5}$$

for each $x \in \Omega$, $c \in \mathbb{R}^N$. Denoting the left-hand side of (5) by $g(x, c)$, we can write (5) as

$$g(x, c) = f(x, cM(x)), \tag{6}$$

where $M(x)$ is the $Nx(N+1)$ matrix defined by

$$M(x) = \begin{pmatrix} \phi_1(x) & \nabla\phi_1(x) \\ & \vdots & \\ \phi_N(x) & \nabla\phi_N(x) \end{pmatrix}. \tag{7}$$

So our task will be to choose $\phi_1(\cdot), \dots, \phi_N(\cdot) \in X$ such that for any H, one can find an f such that the identity (6) holds true. The form of (6) suggests the following way to achieve this aim. Suppose that $f = f(x, y)$ does not depend on u. Then (6) reads

$$g(x, c) = f(x, cJ(x)),$$

where

$$J(x) = \begin{pmatrix} \nabla\phi_1(x) \\ \vdots \\ \nabla\phi_N(x) \end{pmatrix}.$$

Thus if $J(x)$ were a regular matrix at each $x \in \overline{\Omega}$, we could simply define

$$f(x, y) := g(x, yJ^{-1}(x)).$$

Of course, this attempt fails at the fact that, because of the Neumann boundary condition, $J(x)$ cannot be regular at $x \in \partial\Omega$. However, one can achieve that at each $x \in \overline{\Omega}$ one of the $N \times N$ submatrices of $M(x)$ is regular. Then one can use the above definition of f, with $J(x)$ replaced by some submatrix of $M(x)$, at a finite number of different regions, and then match everything together by a partition of unity. This is the outline of the proof which we carry out below. Note that a similar idea is not applicable to Dirichlet boundary condition (3).

For if $\phi_1, \dots, \phi_N$ satisfy $\phi_i|_{\partial\Omega} = 0$, then the matrix $M(x)$ has rank at most 1 at any $x \in \partial\Omega$.

The proof of the theorem consists of the following two steps.

Step 1. We prove that if $\phi_1, \dots, \phi_N \in X \cap C^3(\overline{\Omega})$ are such that the matrix $M(x)$ defined by (7) has rank N at each $x \in \overline{\Omega}$, then the assertion of the theorem holds for the ϕ_i.

Step 2. We find functions $\phi_1, \dots, \phi_N$ satisfying the hypothesis of Step 1.

Proof of Step 1. For $i = 0, \dots, N$, let $M_i(x)$ denote the $N \times N$ matrix obtained from $M(x)$ by omitting the $(i+1)$th column. Let

$$\Omega_i = \{x \in \overline{\Omega}\colon \det M_i(x) \neq 0\}.$$

Since $M(x)$ has rank N everywhere, at each $x \in \overline{\Omega}$ one of the matrices $M_i(x)$ has nonzero determinant. Hence Ω_i, $i = 1, \dots, N$, is a covering of $\overline{\Omega}$. Let $\rho_i(x)$, $i = 1, \dots, N$, be a smooth partition of unity on $\overline{\Omega}$ subordinate to this covering (see [Hi]). (Note that $\overline{\Omega}$ is a smooth manifold with boundary and, by continuity, the Ω_i are open in $\overline{\Omega}$.)

Now, let $H = (h_1, \dots, h_N)\colon \mathbb{R}^N \to \mathbb{R}^N$ be an arbitrary C^1-function and let $g(x, c)$ denote the left-hand side of (5). We define a function f such that (6) holds true.

For any $x \in \overline{\Omega}$ and $(u, y) \in \mathbb{R}^{N+1}$ put

$$f(x, u, y) = \sum_{i=1}^{N} f_i(x, u, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_N) + f_0(x, y_1, \dots, y_N),$$

where

$$f_i(x, z) = f_i(x, z_1, \dots, z_N) = \begin{cases} 0, & \text{if } x \notin \Omega_i, \\ \rho_i(x) g(x, z M_i^{-1}(x)), & \text{if } x \in \Omega_i. \end{cases}$$

($M_i^{-1}(x)$ is the inverse matrix to $M_i(x)$). Since $g \in C^1(\overline{\Omega} \times \mathbb{R}^N)$, $M_i^{-1}(x)$ is of class C^2 in $x \in \Omega_i$ (because the ϕ_i are of class C^3), and ρ_i is smooth with compact support in Ω_i, we have $f \in C^1(\overline{\Omega} \times \mathbb{R}^N)$. Further,

$$f(x, cM(x)) = \sum_{i=0}^{N} f_i(x, cM_i(x)) = \sum_{i=0}^{N} \rho_i(x) g(x, c) = g(x, c),$$

i.e., (6) holds for any $x \in \overline{\Omega}$, $c \in \mathbb{R}^N$. This completes Step 1.

Proof of Step 2. First observe that it is sufficient to find $\phi_i \in C_b^1(\overline{\Omega}) = \{u \in C_1(\overline{\Omega})\colon u$ satisfies (2)$\}$ such that the rank-N property

(r-N) $$\operatorname{rank} M(x) = N \quad \text{for any } x \in \overline{\Omega}$$

holds. Indeed, (r-N) is an open property in the C^1-topology. In other words, if $\phi_1, \dots, \phi_N \in C_b^1(\overline{\Omega})$ have the property (r-N), then so do any sufficiently small C^1-perturbations $\psi_1, \dots, \psi_N$. Now since $C_b^3(\overline{\Omega})$ is dense in $C_b^1(\overline{\Omega})$, one can choose these perturbations in $C_b^3(\overline{\Omega})$.

Below, functions ϕ_i will be defined as the coordinate functions $\phi_i(x) = x_i$ and modified near $\partial\Omega$. For this we introduce functions $d(x)$, $\Pi(x)$ which define the normal coordinates near $\partial\Omega$. Let

$$d(x) := \pm\operatorname{dist}(x, \partial\Omega), \quad - \text{ if } x \in \Omega, \ + \text{ if } x \notin \Omega,$$

and

$$B_r(\partial\Omega) := \{x \in \mathbb{R}^N : |d(x)| < r\}.$$

For $x \in B_r(\partial\Omega)$ let $\Pi(x)$ denote the point of $\partial\Omega$ nearest to x. If $r > 0$ is sufficiently small, then $\Pi(x)$ is well defined and both

$$d(x) : B_r(\partial\Omega) \to (-r, r) \quad \text{and} \quad \Pi(x) : B_r(\partial\Omega) \to \partial\Omega$$

are of class C^2 (any class C^k can be achieved, by making r smaller). We now list some other properties of $d(x)$ and $\Pi(x)$, which will be used below. For the proofs we refer the reader to [He2, G-T]. One has

$$\Pi(x) = x, \qquad d(x) = 0, \qquad \text{for } x \in \partial\Omega, \tag{8}$$

and

$$|\nabla d(x)| = 1 \quad \text{in } B_r(\partial\Omega), \tag{9}$$

where $|\cdot|$ is the Euclidean norm. For any $\delta \in [0, r)$, the set

$$\Omega_\delta := \{x \in \mathbb{R}^N : d(x) < -\delta\}$$

is a subdomain of Ω with the C^2 bundary

$$\partial\Omega_\delta = \{x \in \Omega : d(x) = -\delta\}. \tag{10}$$

The boundaries $\partial\Omega_\delta$ and $\partial\Omega$ are "parallel" in the sense that $\partial\Omega_\delta$ is a constant section of the normal bundle on $\partial\Omega$:

$$\partial\Omega_\delta = \{x - \delta n(x) : x \in \partial\Omega\}. \tag{11}$$

For any $x \in \partial\Omega_\delta$, $\nabla d(x)$ is the unit normal vector to $\partial\Omega_\delta$ pointing out of Ω_δ. In particular,

$$\nabla d(x) = n(x) \quad \text{for } x \in \partial\Omega. \tag{12}$$

Further

$$\Pi(x - \delta n(x)) = \Pi(x) \tag{13}$$

and

$$\nabla d(x - \delta n(x)) = \nabla d(x), \qquad \text{for any } x \in \partial\Omega. \tag{14}$$

We now define functions $\phi_i(x)$ by

$$\begin{aligned} \Phi(x) &:= (\phi_1(x), \dots, \phi_N(x)) \\ &= \begin{cases} x, & \text{for } x \in \Omega_\delta, & (15)\\ \Pi(x) + \beta(d(x))\nabla d(x), & \text{for } x \in \overline{\Omega} \setminus \Omega_\delta, & (16) \end{cases} \end{aligned}$$

where $\delta \in (0, r)$ and $\beta : [-\delta, 0] \to \mathbb{R}$ is a smooth function with the following properties

(i) $\beta(-\delta) = -\delta$,
(ii) $\beta'(-\delta) = 1$,
(iii) $\beta'(0) = 0$,
(iv) $\beta(0) \in (0, \delta)$,
(v) $\beta' > 0$ in $[-\delta, 0)$.

We prove that if δ is sufficiently small (as will be specified later), then the ϕ_i have the required properties.

First we check that the ϕ_i are C^1-functions. This is assured by (i), (ii). Indeed, by (i), (11), (13), and (14), the functions in (15) and (16) coincide on $\partial\Omega_\delta$ and therefore they have the same derivatives in the direction tangent to $\partial\Omega_\delta$. Using (ii), (9), (13), (14), we find that these functions also have the same derivatives in the normal direction $\nabla d(x)$, $x \in \partial\Omega$ (notice that (9) implies $\nabla^2 d(x)[\nabla d(x), \cdot] = 0$). It follows that $\phi_i \in C^1(\overline{\Omega})$.

Next, (iii) implies that the ϕ_i satisfy the Neumann boundary condition.

It remains to verify the rank-N condition (r-N). It is obvious that $M(x)$ has rank N for any $x \in \Omega_\delta$. For the other case, $x \in \overline{\Omega} \setminus \Omega_\delta$, we first calculate the derivative of the mapping $x \mapsto \Phi(x)$. For any vector $v \in \mathbb{R}^N$ we have

$$(17) \quad (D\Phi(x))v = (D\Pi(x))v + \beta'(d(x))\langle\nabla d(x), v\rangle\nabla d(x) + \beta(d(x))(\mathscr{H}(x)v),$$

where $\langle\cdot, \cdot\rangle$ is the usual scalar product on $\mathbb{R}^N$ and $\mathscr{H}(x)$ is the operator on $\mathbb{R}^N$ whose matrix (in the standard basis) is $\nabla^2 d(x)$. Obviously, $\mathscr{H}(x)$ is selfadjoint and, by (9),

$$\mathscr{H}(x)(\nabla d(x)) = 0 \quad \text{for any } x \in B_r(\partial\Omega_\delta).$$

It follows that the range $R(\mathscr{H}(x))$ of $\mathscr{H}(x)$ is orthogonal to $\nabla d(x)$:

$$R(\mathscr{H}(x)) \subset V_x := \{\nabla d(x)\}^\perp.$$

By (14), we have

$$V_x := \{\nabla d(\Pi(x))\}^\perp,$$

i.e., V_x is the tangent space to $\partial\Omega$ at $\Pi(x)$.

We now prove that $D\Phi(x)$ is surjective for any $x \in \Omega \setminus \Omega_\delta$, which implies that $M(x)$ has rank N for any such x.

Realizing that $D\Pi(x)$ is the orthogonal projection onto V_x, we find an $\varepsilon > 0$ such that the restriction

$$(D\Pi(x) + \alpha\mathscr{H}(x))|_{V_x} \colon V_x \to V_x$$

is an isomorphism for any $x \in B_r(\partial\Omega) \cap \overline{\Omega}$ and any $|\alpha| < \varepsilon$. The fact that ε can be chosen independent of x follows from the continuity of $\mathscr{H}(x)$ on the compact set $\overline{B_r(\partial\Omega)}$ (possibly after making r smaller). Thus if $0 < \delta < \min\{\varepsilon, r\}$, then $|\beta(t)| < \varepsilon$ for each $t \in [-\delta, 0]$ (see (iv), (v)) and therefore

$$D\Phi(x)|_{V_x} = (D\Pi(x) + \alpha\mathscr{H}(x))|_{V_x}$$

is an isomorphism onto V_x. Since we further have

$$D\Phi(x)(\nabla d(x)) = \beta'(d(x))\nabla d(x),$$

we conclude, taking (v) into account, that for $x \in \Omega \setminus \Omega_\delta$ the range $R(D\Phi(x))$ contains V_x, as well as the orthogonal space $\operatorname{span}\{\nabla d(x)\}$. This shows that $D\Phi(x)$ is surjective. Consequently, the Jacobi matrix of the functions $\phi_1, \dots, \phi_N$ has rank N, therefore $M(x)$ has rank N for each $x \in \Omega \setminus \Omega_\delta$.

We are left with the case $x \in \delta\Omega$. For such an x, we have, by (iii) and (17), that $D\Phi(x)$ is selfadjoint and $R(D\Phi(x)) = V_x$. Clearly, $M(x)$ has rank N if the vector $\Phi(x)$ does not belong to V_x. But this is indeed the case, because, by (8), (12), (iv), and (4), we have

$$\begin{aligned}\langle \Phi(x), n(x)\rangle &= \langle \Phi(x), \nabla d(x)\rangle = \langle \Pi(x), \nabla d(x)\rangle + \beta(0) \\ &= \langle x, n(x)\rangle + \beta(0) > 0.\end{aligned}$$

We have proved that with the above choice of δ and $\phi_1, \dots, \phi_N$, the property (r-N) holds true. The proof of the theorem is complete.

Remarks. (1) Note that the last inequality is the only place where starshapedness of Ω has been used (referring to (4)). One can easily see that the above construction still comes through if Ω is merely "close" to a starshaped domain. We did not try to find a general construction to include any smooth domain. It would have to be more involved and, yet, the result would not be a significant extension. Presence of complicated dynamics for equations on simple domains (starshaped, convex) is certainly more interesting.

(2) The above construction of the ϕ_i is much simplier if Ω is the ball $\{x: |x| < 1\}$. The reader can easily verify that the functions

$$\phi_i(x) = \beta(|x|)\frac{x_i}{|x|}, \qquad i = 1, \dots, N,$$

where β is a smooth function satisfying

$$\begin{aligned}&\beta(r) \equiv r, && \text{for } r \in [0, 1/2], \\ &\beta'(r) > 0, && \text{for } r \in [0, 1), \\ &\beta'(1) = 0,\end{aligned}$$

have all the required properties.

References

[Am] H. Amann, *Existence and regularity for semilinear parabolic evolution equations*, Scuola Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), 593–696.

[An] S. B. Angenent, *The Morse-Smale property for a semilinear parabolic equation*, J. Differential Equations **62** (1986), 427–442.

[F-M] B. Fiedler and J. Mallet-Paret, *A Poincaré-Bèndixson theorem for scalar reaction diffusion equations*, Arch. Rational Mech. Anal. **107** (1989), 325–345.

[F-P] B. Fielder and P. Poláčik, *Complicated dynamics of scalar reaction diffusion equations with a nonlocal term*, Proc. Roy. Soc. Edinburgh **115A** (1990), 167–192.

[G-T] D. Gilbarg and Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 1977.

[G-H] J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Springer-Verlag, New York, 1983.

[He1] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math., vol. 840, Springer-Verlag, New York, 1981.

[He2] ——, *Perturbation of the boundary value problems for partial differential equations*, Seminairo Brasileiro de Analise, Trabalhos Apresentados Nr. 22, 1985.

[He3] ——, *Some infinite-dimensional Morse-Smale systems defined by semilinear parabolic equations*, J. Differential Equations **59** (1985), 165–205.

[Hi] M. W. Hirsch, *Differential topology*, Springer-Verlag, New York, 1986.

[Ma] H. Matano, *Convergence of solutions of one-dimensional semilinear parabolic equations*, J. Fac. Sci. Univ. Tokyo **30** (1984), 221–227.

[Po1] P. Poláčik, *Convergence in smooth strongly monotone flows defined by semilinear parabolic equations*, J. Differential Equations **79** (1989), 89–110.

[Po2] ——, *Complicated dynamics in scalar semilinear parabolic equations in higher space dimension*, J. Differential Equations **89** (1991), 244–271.

[Si] L. P. Šilnikov, *A contribution to the problem of the structure of an extended neighborhood of a structurally stable equilibrium of the saddle-focus type*, Math. USSR-Sb. **10** (1970), 91–102.

[Ze] T. J. Zelenyak, *Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable*, Differential Equations **4** (1968), 17–22 (translated from Differentsial'nye Uravneniya).

INSTITUTE OF APPLIED MATHEMATICS, COMENIUS UNIVERSITY, MLYNSKA DOLINA 842 15 BRATISLAVA, CZECHOSLOVAKIA