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POSITIVITY OF GLOBAL BRANCHES OF FULLY NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

TIMOTHY J. HEALEY AND HANSJÖRG KIELHÖFER

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ABSTRACT. We consider a bifurcation problem for a general class of fully nonlinear, second-order elliptic equations on a regular bounded domain in $\mathbb{R}^n$ and subject to homogeneous Dirichlet boundary data. We assume that the linearized problem about the trivial solution possesses a positive solution for at least one isolated parameter value. With no other growth or sign conditions imposed upon the nonlinearity, we establish the existence of a global branch of nontrivial positive solutions. Moreover, if there is only one such isolated value of the parameter, we deduce that the branch of positive solutions is unbounded.

1. INTRODUCTION

The model equation,

$$\begin{aligned} \Delta u + \lambda f(u) &= 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \ \lambda \in \mathbb{R}, \\ u &= 0 \quad \text{on } \partial\Omega, \\ u &> 0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

has been studied by many authors; we refer to the reviews [1, 9] and the monograph [12] where many additional references can be found. Besides growth conditions on f near infinity, an essential assumption in most contributions is that the function f is nonnegative on $\mathbb{R}_+$. References [7, 10] are exceptional in this respect in that no sign condition is imposed upon the nonlinearity. Their methods, however, do not allow for the dependence of f on the gradient of u; by adding (and subtracting) some $\lambda cu > 0$, they make f nonnegative in order to apply a minimum principle and a Hopf boundary lemma. The results in [10] appear to be the most general, applying to quasi-linear equations. (We do not deny that in many papers the goal is beyond mere existence of positive branches; the number of positive solutions and the qualitative analysis of bifurcation diagrams are of special interest.)

Here we present a result for a class of nonlinear, elliptic problems over a

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regular domain of the form

$$(1.2)\qquad \begin{aligned} F(\nabla^2 u, \nabla u, u, \mathbf{x}, \lambda) &= 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \ \lambda \in \mathbb{R}, \\ u &= 0 \quad \text{on } \partial\Omega, \\ u &> 0 \quad \text{on } \Omega. \end{aligned}$$

Assuming $F(0, 0, 0, \mathbf{x}, \lambda) = 0$, we have the trivial solution $(\lambda, u) = (\lambda, 0)$ for all $\lambda \in \mathbb{R}$, and we assume that the linearized problem has a positive "eigenfunction" spanning a one-dimensional kernel for some isolated value of the parameter $\lambda = \lambda_0$. For a large class of problems, such a positive eigenfunction is guaranteed by the Krein-Rutman Theorem. We emphasize that, due to the generalization in [7], the zero-order coefficient $F_u(0, 0, 0, \mathbf{x}, \lambda)$ need not be definite in order to obtain a positive eigenfunction. The existence of global bifurcating branches of solutions of fully nonlinear elliptic problems was recently established in [3]. The basic idea is first to differentiate the equation, thus yielding a higher-order quasilinear problem, for which recent generalizations of Rabinowitz's classical theorem [10] are applicable, cf. [3, 8]. In §2 we summarize the relevant steps leading to the existence of a global branch (continuum) of solutions of $(1.2)_{1,2}$. Our main contribution is presented in §3; with *no growth or sign conditions* imposed upon F, we show that all solutions contained in the branch constructed in §2 are *positive* (satisfy $(1.2)_3$). The basic tool here is a minimum principle, as given in [4, 11], for linear equations. If λ_0 happens to be the only parameter value for which the linearized problem admits a positive eigenfunction, we immediately conclude that our positive global solution branch is unbounded in some appropriate norm.

2. Global bifurcation

Let $\Omega \subset \mathbb{R}^n$ be any bounded domain with a boundary $\partial\Omega$ of class $C^{2+\alpha}$, $0 < \alpha < 1$. We study the parameter-dependent boundary value problem,

$$(2.1)\qquad \begin{aligned} F(\nabla^2 u, \nabla u, u, \mathbf{x}, \lambda) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where F is a C^3-function of its arguments, ∇u denotes the gradient with components u_{x_i}, and $\nabla^2 u$ denotes the second gradient or Hessian with components $u_{x_i x_j}$, $i, j = 1, \dots, n$. (Here and in the sequel we use summation convention, and $u_{x_i} \equiv \partial u / \partial x_i$, $u_{x_i x_j} = \partial^2 u / \partial x_i \partial x_j$.) For F we assume *uniform ellipticity*:

(2.2) $\alpha|\xi|^2 \le F_{w_{ij}}(\mathbf{W}, \mathbf{v}, u, \mathbf{x}, \lambda)\xi_i\xi_j \le \beta|\xi|^2$ for all $(\mathbf{W}, \mathbf{v}, u, \mathbf{x}, \lambda) \in \mathbb{R}^{n\times n}_{\text{sym}} \times \mathbb{R}^n \times \mathbb{R} \times \overline{\Omega} \times \mathbb{R}$, $\xi \in \mathbb{R}^n$. The positive constants α, β are uniform on bounded subsets of $\mathbb{R}^{n\times n}_{\text{sym}} \times \mathbb{R}^n \times \mathbb{R} \times \overline{\Omega} \times \mathbb{R}$.

Assuming $F(0, 0, 0, \mathbf{x}, \lambda) = 0$ for all $(\mathbf{x}, \lambda) \in \overline{\Omega} \times \mathbb{R}$, we have the *trivial solution* $(\lambda, u) = (\lambda, 0)$ for all $\lambda \in \mathbb{R}$. For bifurcation from this trivial branch, we study the linearization

(2.3) $F_{w_{ij}}(0, 0, 0, \mathbf{x}, \lambda)u_{x_i x_j} + F_{v_i}(0, 0, 0, \mathbf{x}, \lambda)u_{x_i} + F_u(0, 0, 0, \mathbf{x}, \lambda)u = 0$ in Ω, together with the boundary condition $u = 0$ on $\partial\Omega$.

By (2.2) this is a linear elliptic boundary value problem. A necessary condition for bifurcation at some $(\lambda_0, 0)$ is that (2.3) have a nontrivial solution for $\lambda_0 \in \mathbb{R}$. If the parameter λ occurs only linearly in the zero-order term, then (2.3) is an eigenvalue problem in the classical sense. It is not the goal of this work to study the linear problem (2.3) in its generality, but we simply *assume* that,

(2.4) for some $\lambda_0 \in \mathbb{R}$, the linear operator defined by the left side of (2.3) has a one-dimensional kernel spanned by a function u_0 that is *positive* in Ω, and a closed range of codimension one.

For a large class of problems, the Krein-Rutman Theorem guarantees the existence of u_0 (e.g. see [12]). For a generalization of the classical results, we refer to [7].

The functional analytic setting of (2.1) is described as follows: For $k \in \mathbb{N} \cup \{0\}$, let

(2.5) $C^{k+\alpha}(\overline{\Omega})$ be the usual Hölder spaces with norm $\| \ \|_{k+\alpha}$, $E = C^{\alpha}(\overline{\Omega})$, $D = C^{2+\alpha}(\overline{\Omega}) \cap \{u|_{\partial\Omega} = 0\}$.

Then the left-side of (2.1) defines a mapping

(2.6) $G\colon \mathbb{R} \times D \to E$, which is twice continuously Fréchet differentiable.

The first-order derivative $G_u(\lambda, u)\colon D \to E$ is of the form

$$G_u(\lambda, u)h = F_{w_{ij}}(\nabla^2 u, \nabla u, \mathbf{x}, \lambda)h_{x_i x_j} + F_{v_i}(\nabla^2 u, \nabla u, u, \mathbf{x}, \lambda)h_{x_i} + F_u(\nabla^2 u, \nabla u, \mathbf{x}, \lambda)h \quad \text{for } (\lambda, u) \in \mathbb{R}\times D,\ h \in D, \tag{2.7}$$

By virtue of (2.2), $G_u(\lambda, u)$ is a uniformly elliptic operator for each $(\lambda, u) \in \mathbb{R} \times D$. The Schauder estimate then implies that $G_u(\lambda, u)$ is semi-Fredholm. By assumption (2.4), $G_u(\lambda_0, 0)$ is Fredholm of index zero. Thus, $G_u(\lambda, u)$ has index zero, by the stability of the Fredholm index. Furthermore, it is well known that the operator $G_u(\lambda, u)$ is sectorial and that its spectrum consists of isolated eigenvalues of finite (algebraic) multiplicity. (For more details and a reference, see [5] and [6].) Therefore, by (2.4),

(2.8) zero is an isolated eigenvalue of finite (algebraic) multiplicity of the Fredholm operator $G_u(\lambda_0, 0)\colon D \to E$, where $G_u(\lambda_0, 0)$ is given by the left side of (2.3).

Therefore, the *crossing number* $\chi(\lambda_0)$ through 0 at $\lambda = \lambda_0$ is defined for the family $G_u(\lambda, 0)$ (see [8]), and we assume that

$$\chi(\lambda_0) \text{ is odd.} \tag{2.9}$$

($\chi(\lambda_0)$ is the number of eigenvalues (counting multiplicities) in the 0-group of the eigenvalue perturbation of $G_u(\lambda, 0)$ that leave the left complex halfplane through 0 when the parameter λ passes through λ_0.)

We emphasize that if the Krein-Rutman Theorem is applicable (including the generalization given in [7]), then the zero eigenvalue of $G_u(\lambda_0, 0)$ is simple, and therefore $\chi(\lambda_0) = \pm 1$. In the case of "simplicity in the sense of Crandall-Rabinowitz" [2], their nondegeneracy condition is equivalent to an odd crossing number (see [8]). We do not pursue the problem of odd crossing numbers for $G_u(\lambda, 0)$ near $\lambda = \lambda_0$ in its generality. In particular cases, e.g., [6], $\chi(\lambda_0)$ can be computed explicitly.

As shown in [8], an odd crossing number implies local bifurcation. Therefore,

(2.10) the component $\Sigma_0 \subset \mathbb{R}\times D$ in the closure of the set of nontrivial solutions of $G(\lambda, u) = 0$ that contains $(\lambda_0, 0)$ is not empty.

We call the component Σ_0 *global*, which we now describe in more detail. Since F is fully nonlinear, it does not necessarily define a proper map G (2.6). Therefore we do not know if Σ_0 is subject to the Rabinowitz alternative in $\mathbb{R} \times D$ (see [10] or [8]). There is a way, however, to give a weaker alternative for Σ_0. Following [3], we convert equations (2.1) into a *quasi-linear* elliptic boundary value problem of *fourth order* that has precisely the same solution set as (2.1). As shown in [10] or [8], quasi-linear elliptic operators are proper (and admissible in the sense of [8]; see [6]), and therefore global bifurcating continua are subject to the Rabinowitz alternative, which, in our particular case, reads as follows:

(2.11) Let F be a C^5-function of its arguments. Then Σ_0, emanating at $(\lambda_0, 0)$, is unbounded in $\mathbb{R} \times C^{4+\alpha}(\overline{\Omega})$ or it meets the trivial solution set at some different $(\lambda_1, 0)$.

Remark. There is a price to pay for the above conclusion and also for the more general results of [3], which is not explicitly discussed in that work; namely, the alternative (2.11) does not exclude that $\Sigma_0 \cap (\mathbb{R} \times \{0\}) = (\lambda_0, 0)$ *and* Σ_0 is bounded in $\mathbb{R} \times D$. If (1.2) is quasi-linear, however, then the $C^{4+\alpha}(\overline{\Omega})$-topology in (2.11) can be replaced by the $C^{1+\alpha}(\overline{\Omega})$-norm. Indeed, there is no need to differentiate (1.2) in this case, and we obtain (2.11) directly in terms of the $C^{2+\alpha}(\overline{\Omega})$-topology. Then by the Schauder estimate, boundedness in the $\mathbb{R} \times C^{1+\alpha}$-norm implies boundedness in the $\mathbb{R} \times C^{2+\alpha}$-norm; also see [6].

3. Positivity of the global continuum

Let

(3.1) $P_+ = \{u \in D,\ u > 0 \text{ on } \Omega,\ \partial u/\partial s > 0 \text{ on } \partial\Omega\}$, which is a positive cone in D; here s denotes any direction that enters Ω transversally.

Obviously, P_+ is open in D, and we claim that

$$\Sigma_0 \cap (\mathbb{R} \times P_+) \neq \varnothing. \tag{3.2}$$

Indeed, $u_0 \in P_+$ by Lemma H of [4]. By the Lyapunov-Schmidt reduction, any solution of $G(\lambda, u) = 0$ near $(\lambda_0, 0)$ is of the form

$$u = \varepsilon u_0 + o(\varepsilon) \quad \text{as } \varepsilon \to 0, \tag{3.3}$$

where (λ, ε) satisfies a scalar bifurcation equation

(3.4) $\Phi(\lambda, \varepsilon) = 0$, for which $\Phi_\varepsilon(\lambda, 0)$ changes sign at $\lambda = \lambda_0$ (see [8]).

(As a matter of fact, the change of sign of $\Phi_\varepsilon(\lambda, 0)$ at $\lambda = \lambda_0$ is equivalent to an odd crossing number $\chi(\lambda_0)$ of the family $G_u(\lambda, 0)$ at $\lambda = \lambda_0$.) The Intermediate Value Theorem guarantees a (local) continuum $\{(\lambda, \varepsilon)\}$ near $(\lambda_0, 0)$ with $\varepsilon > 0$ (and $\varepsilon < 0$), and hence (3.2) follows from (3.3) and the assumption (2.4) on u_0.

Assume that Σ_0 leaves $\mathbb{R} \times P_+$. By the connectedness of Σ_0, there must be a solution $(\lambda^*, u^*) \in \Sigma_0$ belonging to the boundary of $\mathbb{R} \times P_+$: that is,

(3.5) $u^* \geq 0$ on $\overline{\Omega}$, and there is at least one point $\mathbf{x}_0 \in \overline{\Omega}$ such that $u^*(\mathbf{x}_0) = \nabla u^*(\mathbf{x}_0) = 0$.

Observe that (λ^*, u^*) solves the *linear* partial differential equation

$$
\begin{aligned}
L^*u = {} & \left[\int_0^1 F_{w_{ij}}(t\nabla^2 u^*, t\nabla u^*, tu^*, \mathbf{x}, \lambda^*)\,dt\right] u_{x_i x_j} \\
& + \left[\int_0^1 F_{v_i}(t\nabla^2 u^*, t\nabla u^*, tu^*, \mathbf{x}, \lambda^*)\,dt\right] u_{x_i} \\
& + \left[\int_0^1 F_u(t\nabla^2 u^*, t\nabla u^*, tu^*, \mathbf{x}, \lambda^*)\,dt\right] u = 0 .
\end{aligned} \tag{3.6}
$$

As first observed by Serrin [11], the usual minimum principle for elliptic equations also holds *without* restricting the sign of the coefficient of the zero-order term, provided that $u^* \geq 0$ (cf. also [4]). We employ this result, including the Hopf boundary lemma ([4, Lemma H]), to conclude:

$$\text{Condition (3.5) implies } u^* \equiv 0 . \tag{3.7}$$

In other words,

$$\text{if } (\lambda^*, u^*) \in \Sigma_0 \text{ is on the boundary of } \mathbb{R} \times P_+ \text{ then } u^* \equiv 0 . \tag{3.8}$$

The same arguments employed above (maximum principle) imply the existence of a global *negative* part of Σ_0 that cannot leave $\mathbb{R} \times P_-$ unless $u^* \equiv 0 (P_- \equiv -P_+)$. We summarize:

Theorem 3.1. *Assume the hypotheses of* §2. *Then the global continuum Σ_0 bifurcating at $(\lambda_0, 0)$ consists of a positive part in P_+ and a negative part in P_-, each subject to the alternative* (2.11).

Remark. Since we have not assumed oddness of $G(\lambda, \cdot)$, we do not exclude the possibility that the two parts behave differently.

Finally we note that

Corollary 3.1. *Assume the hypotheses of* §2. *If $\lambda_0 \in \mathbb{R}$ is the only parameter value for which* (2.3) *admits a positive eigenfunction, then Σ_0 is unbounded in $\mathbb{R} \times C^{4+\alpha}(\overline{\Omega})$.*

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Department of Theoretical and Applied Mechanics and Center for Applied Mathematics, Cornell University, Ithaca, New York 14853-1503

Institut für Mathematik, Universität Augsburg, Universitätsstrasse 8, 8900 Augsburg, Germany