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GAUSSIAN PERIODS AND UNITS IN CERTAIN CYCLIC FIELDS

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ABSTRACT. We analyze the property of period-unit integer translation (there exists a Gaussian period η and rational integer c such that $\eta + c$ is a unit) in simplest quadratic, cubic, and quartic fields of arbitrary conductor. This is an extension of work of E. Lehmer, R. Schoof, and L. C. Washington for prime conductor. We also determine the Gaussian period polynomial for arbitrary conductor.

1. INTRODUCTION

In [10], Lehmer exhibited a remarkable property of the simplest cubic fields of Shanks [13] and the simplest quartic fields of Gras [4]. These fields, which were already known for their small regulators, are defined by the polynomials

$$\begin{aligned} S_3(X) &= X^3 - tX^2 - (t+3)X - 1\,, \qquad && t \in \mathbb{Z},\ t \geq -1\,; \\ S_4(X) &= X^4 - tX^3 - 6X^2 + tX + 1\,, \qquad && t \in \mathbb{Z}^+,\ t \neq 3\,. \end{aligned}$$

When $p = t^2 + 3t + 9$ (respectively, $p = t^2 + 16$) is prime, it is the conductor of the field. Lehmer showed that in this case the roots, which are units, differ by a rational integer from the classical cubic and quartic Gaussian periods, whose minimal polynomials are

$$\Psi_3(X) = X^3 + X^2 - \frac{p-1}{3}X - \frac{(a+3)p-1}{27}, \qquad a = \pm(2t+3)\,,\ a \equiv 1 \bmod 3\,,$$

$$\begin{aligned} \Psi_4(X) = X^4 + X^3 &- \frac{3(p-1)}{8}X^2 + \frac{2ap - 3p + 1}{16}X \\ &+ \frac{p^2 - 4a^2p + 8ap - 6p + 1}{256}, \qquad a = \pm t\,,\ a \equiv 1 \bmod 4\,, \end{aligned}$$

where p is the conductor. (The somewhat inconsistent normalization of a and t is retained from the literature.) We will say that fields with a Gaussian period differing by a rational integer from a unit of infinite order possess *Period-Unit Integer Translation* (PUIT).

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Subsequently Schoof and Washington [12] showed that period-unit integer translation almost defines the simplest fields. The cubic fields of prime conductor with this property are precisely the Shanks fields. The quartic fields of prime conductor where $\eta + c$ is a unit of absolute norm 1 for some period η and some $c \in \mathbb{Z}$ are Gras fields.

This paper extends these results to arbitrary conductor. Classical (i.e., prime conductor p) cyclotomy defined the principal cyclotomic class of degree e to be

$$\mathscr{C}_0 = \left\{ g^{ej} \bmod p ,\ j = 0, \dots, \frac{p-1}{e} - 1 \right\},$$

where g is any primitive root $\bmod p$. The principal Gaussian period $\eta_0 = \sum_{j \in \mathscr{C}_0} \zeta^j$, where $\zeta = \zeta_p = \exp(2\pi i/p)$. Since composite conductors do not have a primitive root, we must modify the definition. The natural extension is to embed an abelian field K in $\mathbb{Q}[\zeta_F]$, where F is the conductor of K, and to define $\eta_0 = \mathrm{Tr}_K^{\mathbb{Q}[\zeta_F]} \zeta_F$; this coincides with the classical definition when F is prime. The class $\mathscr{C}_0$ becomes the kernel in $(\mathbb{Z}/F\mathbb{Z})^*$ of the character group associated to K through duality.

2. The quadratic and cubic simplest fields

For completeness we begin with the easy quadratic case. The Möbius function is denoted $\mu(\cdot)$.

Proposition 2.1. *The field $\mathbb{Q}[\sqrt{m}]$ has period-unit integer translation if and only if $m = (2t+1)^2 \pm 4$ for some $t \in \mathbb{Z}^+$.*

Proof. If $m < 0$ all units are torsion. Suppose $m > 0$. The Gaussian period $\eta = \frac{1}{2}(\tau(\chi) + \tau(\chi^2))$, where τ is the Gauss sum and χ is the numerical character with the same conductor as the field $\mathbb{Q}[\sqrt{m}]$. If $m \not\equiv 1 \bmod 4$ then this conductor is $4m$ and η vanishes. For $m \equiv 1 \bmod 4$, Gauss showed [7] that $\tau(\chi) = \sqrt{m}$. For the trivial character χ^2, $\tau(\chi^2) = \mu(m)$. Then the period polynomial is $X^2 - \mu(m)X + \frac{1-m}{4}$. If $\mathbb{Q}[\sqrt{m}]$ has PUIT,

$$\pm 1 = N_{\mathbb{Q}}^{\mathbb{Q}[\sqrt{m}]}(\eta + c) = \frac{(2c + \mu(m))^2 - m}{4} \Rightarrow m = (2c + \mu(m))^2 \pm 4$$

and the proposition follows easily. Reverse the steps to prove the converse. □

The conductor of the field defined by the Shanks polynomial is $F = t^2 + 3t + 9$ whenever this quantity is squarefree or $\gcd(F, 27) = 9$ and $F/9$ is squarefree. We consider only these cases, otherwise we have no control over the extraneous factors in F.

Proposition 2.2. *All Shanks cubic fields with conductor $F = t^2 + 3t + 9$ have PUIT.*

Proof. The minimal polynomial of η_0 is [2]

$$\Psi_3(X) = \begin{cases} X^3 - \mu(F)X^2 - \dfrac{F-1}{3}X + \mu(F)\dfrac{(a+3)F-1}{27}, & 3 \nmid F, \\[2ex] X^3 - \dfrac{F}{3}X - \mu\left(\dfrac{F}{9}\right)\dfrac{aF}{27}, & \text{otherwise}, \end{cases}$$

where $F = \frac{a^2+27b^2}{4}$; $a \equiv b \mod 2$; $a \equiv 1 \mod 3$ in the first case and $a = 3a_0$, $a_0 \equiv 1 \mod 3$ in the second. The Shanks fields are those with $b = 1$. Then $S_3(X + (t - (\frac{t}{3})_2)/3) = \Psi_3(\pm X)$, where $(\div)_2$ is the Jacobi symbol. □

The Schoof-Washington result does not hold in general. The field of conductor 91 corresponding to $(a, b) = (-11, 3)$ is not simplest but

$$\Psi_3(X-1) = X^3 - 4X^2 - 25X + 1.$$

3. The quartic case

To control the conductor, we restrict simplest quartic fields to the Gras polynomials where $t^2 + 16$ is not divisible by an odd square.

Theorem 1. *A simplest quartic field has PUIT if and only if* $t = 4$ *or* $4 \nmid t$.

In the sequel K will be a real cyclic quartic number field of conductor F. (Imaginary cyclic quartic fields are not of interest because the free part of the unit group is generated by the fundamental unit in the quadratic subfield.) Such a field has a unique quadratic subfield k, whose conductor we will call m. The parameter F/m occurs frequently, so we will denote it by G. $\mathrm{Gal}(K/\mathbb{Q}) = \langle\sigma\rangle \cong \mathbb{Z}/4\mathbb{Z}$. $\nu_2(\cdot)$ is a 2-adic valuation. If there exists a unit ε such that -1, ε, and the Galois conjugates of ε generate the full unit group, then ε is called a *Minkowski unit.* If $N_k^K\varepsilon = \varepsilon^{1+\sigma^2} = \pm 1$ then ε is called a *χ-relative unit.*

We have been able to prove only a weak counterpart of the Schoof-Washington theorem.

Theorem 2. *If in a real cyclic quartic field* K, $G = 1$ *and there exist a Gaussian period* η *and a rational integer* c *such that* $\eta + c$ *is a* χ*-relative unit, then* K *is a simplest quartic field.*

The remainder of the paper is devoted to the proof of these results. Recall a Gaussian integer $\alpha = a+bi$ is called *primary* if and only if $\alpha \equiv 1 \mod 2(1+i)$. This is equivalent to the conditions $a+b \equiv 1 \mod 4$ and $b \equiv 0 \mod 2$. Note that b is determined only up to sign. Every Gaussian integer relatively prime to 2 has a unique primary associate. The product of primary numbers is primary [7].

Lemma 3.1 (Hasse [6]). *For* m *odd and* G *fixed, there is a one-to-one correspondence between*

(1) *cyclic quartic fields of conductor* F,
(2) *conjugate pairs of numerical quartic characters of conductor* F,
(3) *representations of* $m = a^2 + b^2$ *where* $a+bi$ *is primary and furthermore* $b > 0$,
(4) *primary Gaussian integers* $\psi = a + bi$ *of norm* m, *up to complex conjugation.*

Proof. The correspondence between (1) and (2) is given by the field K belonging to a character χ. The correspondence between (3) and (4) is given by $N_{\mathbb{Q}}^{\mathbb{Q}[i]}$. The relationship between (1) and (4) is more subtle. The character χ may be factored into prime power components: $\chi = \chi_{p_0}\chi_{p_1}\cdots\chi_{p_r}\xi_G$, where each χ_p

is a quartic residue symbol for primary α_p lying over p in $\mathbb{Z}[i]$ and ξ_G is a quadratic character of conductor G. A primary element ψ of norm m is defined by $\psi = \prod_{j=0}^{r} \alpha_{p_j}$. □

The case m even is handled similarly, except that $\psi = 2(1+i)\psi_0$, $\psi_0 = \prod_{j=1}^{r} \alpha_{p_j}$, with each choice of ψ_0 corresponding to a different field of conductor F.

Define a and b by

$$\psi = a + bi. \tag{3.1}$$

The field is real if the corresponding character χ is even, i.e., $\chi(-1) = 1$. Hasse [6] expressed elements of real cyclic quartic fields in terms of a $\mathbb{Q}$-basis of four elements. The 4-tuple $[x_0, x_1, y_0, y_1]$ represents the number

$$\tfrac{1}{4}(x_0 + \delta x_1\sqrt{m} + (y_0 + iy_1)\tau(\chi) + (y_0 - iy_1)\overline{\tau(\chi)}), \qquad \delta = \pm 1,$$

where $\tau(\chi)$ is the Gauss sum $\sum_{j=1}^{F-1} \chi(j)\zeta_F^j$. The ambiguous sign is determined below. An element is an integer of K if and only if $x_0, x_1, y_0, y_1 \in \mathbb{Z}$ and

$$m \text{ odd } x_0 \equiv x_1, \qquad \frac{x_0 + x_1}{2} \equiv Gy_0, \qquad \frac{x_0 - x_1}{2} \equiv Gy_1 \mod 2,$$
$$m \text{ even } x_0 \equiv 0 \mod 4, \qquad x_1 \equiv 0 \mod 2.$$

Galois action is described easily in this basis:

$$\sigma\colon [x_0, x_1, y_0, y_1] \mapsto [x_0, -x_1, -y_1, y_0].$$

Let χ^* be the purely quartic part of χ; define μ_2 to be $\mu(\text{oddpart }(m))$. (The subscript is selected to emphasize that oddpart is the 2-free-part.) The sign δ is determined at [6, §7(12)]

$$\delta = \begin{cases} \mu_2(\frac{G}{m})_2, & 2 \nmid m, \\ \mu_2\chi^*(-1)(\frac{G/2}{m})_2, & 2 \mid m. \end{cases}$$

Let $\tau = \tau(\chi)$. Multiplication of elements can be accomplished by the table[1]

$$\tau\overline{\tau} = F, \qquad \tau^2 = G\psi\delta\sqrt{m}, \qquad \overline{\tau}^2 = G\overline{\psi}\delta\sqrt{m}, \qquad \sqrt{m}\tau = \delta\psi\overline{\tau}.$$

Write $\alpha = [x_0, x_1, y_0, y_1] \in K\backslash k$. Let $X^4 - c_1X^3 + c_2X^2 - c_3X + c_4$ be the minimal polynomial $\operatorname{Irr}_{\mathbb{Q}}\alpha$. From the relations

$$\begin{aligned} &2(\alpha + \alpha^{\sigma^2}) = x_0 + \delta x_1\sqrt{m}, \\ &16N_k^K\alpha = 16\alpha^{1+\sigma^2} = x_0^2 + mx_1^2 - 2F(y_0^2 + y_1^2) \\ &\qquad + 2\delta\sqrt{m}(x_0x_1 - G(a(y_0^2 - y_1^2) - 2by_0y_1)), \end{aligned} \tag{3.2}$$

we obtain the

Minimal Polynomial Formula (MPF).

$$\begin{aligned} c_1 &= x_0 = \operatorname{Tr}\alpha, \\ 8c_2 &= x_0^2 + mx_1^2 - 2F(y_0^2 + y_1^2) + 2(x_0^2 - mx_1^2), \\ 16c_3 &= x_0(x_0^2 + mx_1^2 - 2F(y_0^2 + y_1^2)) - 2mx_1(x_0x_1 - G(a(y_0^2 - y_1^2) - 2by_0y_1)), \\ 256c_4 &= (x_0^2 + mx_1^2 - 2F(y_0^2 + y_1^2))^2 - 4m(x_0x_1 - G(a(y_0^2 - y_1^2) - 2by_0y_1))^2. \end{aligned}$$

The period polynomial is of independent interest.

[1] This table appears in Hasse [6]. The similar table in Gras [4] does not determine δ; nevertheless, her formula for the minimal polynomial of a generic element is correct.

Theorem 3. *The period polynomial of a real cyclic quartic field of conductor* F *is*

$$\Psi_4(X) = \begin{cases} X^2 - \mu(F)X^3 - \dfrac{3(F-1)}{8}X^2 - \mu(F)\dfrac{2aF-3F+1}{16}X \\ \quad + \dfrac{F^2 - 4a^2F + 8aF - 6F + 1}{256}, & 2 \nmid G, \\ X^4 - \dfrac{F}{4}X^2 + \dfrac{FGb^2}{64}, & 2 \mid G, \end{cases}$$

where a *and* b *are defined by* (3.1).

Proof. In Hasse's basis, $\eta_0 = [\mu(F), \mu(F), 1, 0]$ because

$$\eta_0 = \tfrac{1}{4}(\tau(\chi) + \tau(\chi^2) + \tau(\overline{\chi}) + \tau(\chi^4)),$$
$$\tau(\chi^4) = \mu(F),$$
$$\tau(\chi^2) = \mu(G)(G/m)_2\sqrt{m},$$
$$\delta^{-1}\mu(G)(G/m)_2 = \mu(F).$$

The formula for $\tau(\chi^2)$ can be proved by induction on the number of primes dividing G. The other assertions are elementary. Note particularly that if $2 \| G$ then $2 \mid m$ and $\mu(F) = 0$. The proof is completed by substituting η_0 into MPF. □

The period polynomial for odd G was determined by Nakahara [11]. The period polynomial for F prime was discovered by Gauss [3, §22], who left the proof as an exercise to the reader. The details appear in Bachmann [1].

Caution. Hasse [6] used χ^2 for the induced *primitive* quadratic character.

Lemma 3.2. *The conductors and corresponding primary integer* ψ *of* K *and the largest root* ε *of* $S_4(X)$ *depend on* $\nu_2(t)$ *and are given by*

Case	$\nu_2(t)$	F	G	ψ	ε
1	0	t^2+16	1	$a+4i$	$[\lvert a\rvert, \operatorname{sgn} a, 1, 0]$
2	1	t^2+16	4	$a+2i$	$[2\lvert a\rvert, 2\operatorname{sgn} a, 1, 0]$
3	2	$\frac{t^2+16}{2}$	2	$\pm 2+bi$	$[2b, -2, 1, 1]$
4	≥ 3	$\frac{t^2+16}{2}$	8	$\pm 1+bi$	$[4b, -4, 1, 1]$

Proof. The conductor is obtained in [8, (6)] using the formulas in [5]. ψ is obtained by factoring $m = F/G$ in $\mathbb{Z}[i]$. The tuples for ε are then verified by substitution into MPF.[2] □

Notation. Cases 1–4 retain their meaning from this table throughout.

The proof of Theorem 2 follows almost immediately from these tuples.

Proof. If $\alpha \in E_\chi$ then from (3.2)

$$(x_0x_1 - G(a(y_0^2 - y_1^2) - 2by_0y_1)) = 0. \tag{3.3}$$

[2]The tuples for ε in [4] are not normalized with respect to signs.

Let us consider the case $\mu(F) = 1$. Without loss of generality, we may assume $\eta = \eta_0$ because the periods are Galois conjugates. Fix $G = 1$ and substitute the tuple $\alpha = \eta_0 + c = [1+4c, 1, 1, 0]$ into (3.3). Then $c = \frac{a-1}{4}$. From Theorem 3 and Lemma 3.2 we have $(\operatorname{sgn} a)(\eta + c) = \varepsilon$. The $\mu(F) = -1$ case is handled identically. □

Theorem 1 is an immediate consequence of the more detailed

Proposition 3.3. *Let η be the Gaussian period of a simplest quartic field. For $c \in \mathbb{Z}$*

Case 1. *$\eta + c$ is a χ-relative unit for $c = \mu(F)\frac{a-1}{4}$ and no other values of c, except for $t = 1$, when $\eta - 2$ is a Minkowski unit.*

Case 2. *$\eta + c$ is a unit for $c = \pm 1$ and no other values of c, except for $t = 2$, when $\eta \pm 2$ are also units.*

Case 3. *$\eta + c$ is a unit only for $t = 4$ (the field of conductor* 16*) and $c = \pm 1$. These units are Minkowski.*

Case 4. *$\eta + c$ is never a unit.*

Proof. Since the periods are conjugates, it suffices to consider η_0. It is necessary to break the proof up into the four cases depending on $\nu_2(t)$.

Case 1. Consider first $\mu(F) = -1$. Substituting $\eta_0 + c = [-1+4c, -1, 1, 0]$ into MPF gives the constant term

$$N = c^4 - c^3 + (-6a_0^2 - 3a_0 - 6)c^2 + (-8a_0^3 - 3a_0^2 - 8a_0 + 1)c - 3a_0^4 - a_0^3 - 2a_0^2 + a_0 + 1$$

where $a = 4a_0 + 1$. For $\eta_0 + c$ to be a unit, $N = \pm 1$.

$$\begin{aligned} N = 1 \Rightarrow (c + a_0)(c^3 + (-a_0 - 1)c^2 \\ + (-5a_0^2 - 2a_0 - 6)c - 3a_0^3 - a_0^2 - 2a_0 + 1) = 0. \end{aligned}$$

This factorization was carried out by the symbolic algebra package MACSYMA. The first factor corresponds to the solution $c = \mu(F)\frac{a-1}{4}$. The second factor has no roots $\bmod 2$, hence no roots in $\mathbb{Z}$.

It turns out that $N = -1$ has a unique solution in which $a = t = 1$. Define $\widehat{N} = N + 1$, and consider $\widehat{N}$ as a polynomial with a_0 fixed and c an indeterminate.

$$\begin{aligned} (3.4) \quad \widehat{N} = 0 \Rightarrow c^4 - c^3 + (-6a_0^2 - 3a_0 - 6)c^2 \\ + (-8a_0^3 - 3a_0^2 - 8a_0 + 1)c - 3a_0^4 - a_0^3 - 2a_0^2 + a_0 + 2 = 0. \end{aligned}$$

If $a_0 = 0$, i.e., $a = 1$, this reduces to

$$(c + 2)(c^3 - 3c^2 + 1) = 0.$$

The second factor is irreducible. Since $a = 1 \Rightarrow F = 17$ and $\mu(17) = -1$, the first factor is a valid solution. A MACSYMA calculation shows that $(\eta_0 - 2)^{1+\sigma^2} = 4 - \sqrt{17} = \varepsilon_k^\sigma$, so it is a Minkowski unit by [9, Proposition 3.2]. We claim that this is the only solution to (3.4) for $a_0, c \in \mathbb{Z}$.

Claim. Equation (3.4) has no integer solutions for c for all $a_0 \neq 0$.

Proof. An integer solution of (3.4) corresponds to a root of $\widehat{N}$. Notice that if $a_0 = \pm 1$, $\widehat{N}$ is irreducible, so assume $|a_0| \geq 2$. Our intention is to show that all four roots of $\widehat{N}$ (for some fixed value of a_0) occur between consecutive integers. The roots may be located from the following table of values.

c	$\widehat{N}(c)$	$\lim_{\lvert a_0\rvert\to\infty} \operatorname{sgn}\widehat{N}(c)$
$3a_0$	$-2(32a_0^3+40a_0^2-2a_0-1)$	$-\operatorname{sgn} a_0$
$3a_0+1$	$-(80a_0^2+40a_0+3)$	$-$
$3a_0+2$	$2(32a_0^3+8a_0^2-18a_0-6)$	$\operatorname{sgn} a_0$
$-a_0-2$	$24a_0$	$\operatorname{sgn} a_0$
$-a_0-1$	-3	$-$
$-a_0$	2	$+$
$-a_0+1$	-3	$-$
$-a_0+2$	$-24a_0-12$	$-\operatorname{sgn} a_0$

All roots of the polynomials in the middle column lie in $(-2, 2)$; hence for $|a_0| \geq 2$ the sign of $\widehat{N}(c)$ is given by the third column. $\widehat{N}$ has one irrational root in $(3a_0, 3a_0+2)$ and three irrational roots in $(-a_0-2, -a_0+2)$. This accounts for all four roots.

Applying the same method to the case $\mu(F) = 1$ yields the same results up to sign: either $c = \frac{a-1}{4}$ or $a = 1$. However, since $\mu(17) \neq 1$, $a = 1$ is not a solution. □

Case 2. In Case 2, $G = 2$ and $b = 2$. From MPF,

$$N = N_{\mathbb{Q}}^K(\eta + c) = N_{\mathbb{Q}}^K[4c, 0, 1, 0] = c^4 - (a^2+4)c^2 + a^2 + 4.$$

The author shows in [9] that in Case 2 all units have norm 1 and that the entire unit group is generated by ε of Lemma 3.2 and the quadratic fundamental unit ε_k. Solving $N = 1$ in terms of c we have $c \in \{\pm\sqrt{a^2+3}, \pm 1\}$. The radical is in $\mathbb{Z}$ only for $a = \pm 1$, and since $b \equiv 2 \mod 4$, $a = -1$. This is the one exception noted.

We can solve for $\eta \pm c$ in terms of ε and ε_k. In Hasse's basis, $\varepsilon_k = [2|a|, 2\mu_2, 0, 0]$. From the clue that

$$N_k^K(\eta + 1) = \begin{cases} -\varepsilon_k^2, & a\mu_2 > 0, \\ -\varepsilon_k^{2\sigma}, & a\mu_2 < 0, \end{cases}$$

we find that $\eta_0 + 1 = -\varepsilon_k^{\sigma^{(\operatorname{sgn} a+1)/2}} \varepsilon^{\sigma^{\operatorname{sgn} a}}$. The exceptional unit $\eta + 2$ for $t = 2$ was determined in the same way: $\eta_0 + 2 = \varepsilon_k^{\sigma} \varepsilon^{1+\sigma^{-1}}$.

Case 3.

$$N_{\mathbb{Q}}^K(\eta + c) = 1 \Rightarrow c \in \left\{\pm 2\sqrt{\pm 2\sqrt{b^2+8} + b^2 + 4}\right\},$$

$$N_{\mathbb{Q}}^K(\eta + c) = -1 \Rightarrow c \in \left\{\pm 2\sqrt{b^2 \pm 2b + 4}\right\}.$$

The $N = 1$ case has no possible solutions of $c \in \mathbb{Z}$ since $b > 1$. The only valid b that gives an integer solution for c in the $N = -1$ case is $b = 2$, which corresponds to $t = 4$ and $F = 16$. Therefore in this field $\eta \pm 1$ are units; in fact, they are Minkowski units by [9, Proposition 3.2] because $(\eta_0 + 1)^{1+\sigma^2} = -(1 - \sqrt{2}) = -\varepsilon_k^{\sigma}$.

Case 4.

$$N_{\mathbb{Q}}^{K}(\eta + c) = 1 \Rightarrow c \in \left\{ \pm\sqrt{\pm\sqrt{b^2+2} + b^2 + 1} \right\},$$
$$N_{\mathbb{Q}}^{K}(\eta + c) = -1 \Rightarrow c \in \left\{ \pm\sqrt{b^2 \pm b + 1} \right\}.$$

Neither of these equations can have an integral solution when $b > 1$. □

There is another relationship between periods and units.

Proposition 3.4. *For ε satisfying $S_4(X)$, the following elements are Gaussian periods.*

Case 1. $\mu(F)(\varepsilon \operatorname{sgn} a - \frac{a-1}{4})$.

Case 2. $\frac{\varepsilon - \varepsilon^{\sigma^2}}{2}$.

Cases 3 *and* 4. $\frac{\varepsilon + \varepsilon^{\sigma} - t/2}{2}$.

Proof. Compare the tuples for ε and η. □

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