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ON THE KOSTANT CONVEXITY THEOREM

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ABSTRACT. A quick proof that the coadjoint orbits of a compact connected Lie group project onto convex polytopes in the dual of a Cartan subalgebra.

1. INTRODUCTION

Let G be a compact connected Lie group, T a maximal torus of G, $\mathfrak{g}$ and $\mathfrak{t}$ their Lie algebras and $\pi: \mathfrak{g}^* \to \mathfrak{t}^*$ the natural projection. As usual we identify $\mathfrak{t}^*$ with the subspace of all T-fixed points in $\mathfrak{g}^*$. Then every coadjoint orbit X of G intersects $\mathfrak{t}^*$ in a Weyl group orbit Ω_X [4], and in this setting B. Kostant [9] has proved:

1.1. **Theorem.** *$\pi(X)$ is the convex hull of Ω_X.*

Alternative proofs and generalizations have appeared in [2, 5, 7, 8]; see the monograph [3]. Our purpose here is to show that representation theory and the projective embeddings of Borel-Weil-Tits [10, 12] allow for an elementary proof of Theorem 1.1, bypassing the Morse theoretic or asymptotic arguments of *loc.cit.*

2. PROJECTIVE EMBEDDINGS

If Ω_X lies in the weight lattice $\Lambda = \{w \in \mathfrak{t}^* : e^{i\langle w, Z\rangle} = 1 \ \forall\, Z \in \ker(\exp|\mathfrak{t})\}$, we say that X is *integral*; then Ω_X contains the highest weight w_0 of a unique irreducible unitary G-module V [1]. The corresponding projective space $\mathbf{P}(V)$, regarded as the manifold of all rank one hermitian projectors $\mathbf{p}$ in V, carries canonical complex and symplectic structures J and σ, defined on tangent vectors $\delta\mathbf{p}, \delta'\mathbf{p} \in \mathrm{T}_{\mathbf{p}}\mathbf{P}(V)$ by

$$J\delta\mathbf{p} = \frac{1}{i}[\mathbf{p}, \delta\mathbf{p}], \qquad \sigma(\delta\mathbf{p}, \delta'\mathbf{p}) = \mathrm{Tr}(\delta'\mathbf{p}J\delta\mathbf{p}).$$

Writing $\mathbf{E}_0$ for the eigenprojector associated to w_0, we know from [10, 12] that the G-orbit of $\mathbf{E}_0$ is a *complex* submanifold, $\mathbf{X}$, of $\mathbf{P}(V)$. In particular $\mathbf{X}$ is homogeneous symplectic, with momentum map $\Phi: \mathbf{X} \to \mathfrak{g}^*$ readily computed

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as

$$(*) \qquad \langle \Phi(\mathbf{x}), Z \rangle = \frac{1}{i} \mathrm{Tr}(\mathbf{x}\mathbf{Z}),$$

where $Z \mapsto \mathbf{Z}$ is the differentiated representation of $\mathfrak{g}$ on V. By Kirillov-Kostant-Souriau [11] Φ covers a coadjoint orbit of G, namely X since $\Phi(\mathbf{E}_0) = w_0$. But X is simply connected [10], so Φ is actually a diffeomorphism $\mathbf{X} \leftrightarrow X$.

3. Proof of the theorem

If Theorem 1.1 holds when Ω_X lies in Λ, it follows also for Ω_X in $\mathbf{R}\Lambda$ by rescaling, and then for the general Ω_X in $\mathfrak{t}^* = \overline{\mathbf{R}\Lambda}$ by a straightforward continuity argument. So it is enough to prove Theorem 1.1 when X is integral.

Let, then, $\mathbf{X} \subset \mathbf{P}(V)$ be as above; also let $\Delta \subset \mathfrak{t}^*$ be the weight diagram of V, so that we have

$$\frac{1}{i}\mathbf{Z} = \sum_{w \in \Delta} \langle w, Z \rangle \mathbf{E}_w \quad \forall Z \in \mathfrak{t},$$

where $\mathbf{E}_w$ denotes the eigenprojector belonging to $w \in \Delta$. Substituting this in $(*)$ exhibits $\pi(\Phi(\mathbf{x}))$ as a convex combination of elements of Δ; since Δ lies in the convex hull of Ω_X [1] so does, therefore, $\pi(X)$.

For the converse inclusion we use a variational method inspired from [6]. Let $\{w_j\}$ be an enumeration of Ω_X and write $\mathbf{E}_j$ for the projectors $\Phi^{-1}(w_j) = \mathbf{E}_{w_j}$. Given a convex combination $\sum_j \mu_j w_j$ of the w_j, we maximize the non-negative function

$$\rho(\mathbf{x}) = \prod_j \mathrm{Tr}(\mathbf{E}_j \mathbf{x})^{\mu_j}$$

and compute its derivative $D\rho(\mathbf{x})(\delta\mathbf{x})$ in the tangent direction

$$\delta\mathbf{x} = J[\mathbf{Z}, \mathbf{x}], \qquad Z \in \mathfrak{t}.$$

Since $\mathbf{X}$ is compact ρ does attain its maximum, which is positive: if ρ vanished identically, so would the product of the real analytic functions $\rho_j(\mathbf{x}) = \mathrm{Tr}(\mathbf{E}_j\mathbf{x})$ and hence also one of the ρ_j, whereas $\rho_j(\mathbf{E}_j) = 1$. Now we have

$$\begin{aligned} D\rho_j(\mathbf{x})(\delta\mathbf{x}) = \mathrm{Tr}(\mathbf{E}_j \delta\mathbf{x}) &= \frac{1}{i}\mathrm{Tr}(\mathbf{E}_j[2\mathbf{x}\mathrm{Tr}(\mathbf{x}\mathbf{Z}) - \mathbf{Z}\mathbf{x} - \mathbf{x}\mathbf{Z}]) \\ &= 2\rho_j(\mathbf{x})\langle \Phi(\mathbf{x}) - w_j, Z \rangle, \end{aligned}$$

whence

$$D\rho(\mathbf{x})(\delta\mathbf{x}) = 2\rho(\mathbf{x}) \left\langle \Phi(\mathbf{x}) - \textstyle\sum_j \mu_j w_j, Z \right\rangle = 0 \quad \forall Z \in \mathfrak{t}$$

at the maximum. Thus $\Phi(\mathbf{x})$ projects to the given convex combination, and our proof is complete.

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[1] *Note added in proof*. Michèle Vergne has kindly pointed out that V. G. Kac & D. H. Peterson [13] also used projective embeddings (but not the short variational argument above) to prove Theorem 1.1.

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