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THE ITERATED TOTAL SQUARING OPERATION

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ABSTRACT. In this paper we prove a formula that expresses the iterated total squaring operation in terms of modular invariant theory and provide an alternative proof of a classical result of Múi's.

1. INTRODUCTION

We start by recalling some notation from invariant theory (see [5]). Let

$$P_m = \mathbb{F}_2[t_1, \dots, t_m] \; ;$$

$$e_m = \prod \Big(\sum_{i=1}^{m} \lambda_i t_i\Big) \qquad (\lambda_i = 0, 1 \; , \; \textstyle\sum \lambda_i > 0) \; . \tag{1.1}$$

We set $\Phi_m = P_m[e_m^{-1}]$. The natural action of $\mathrm{GL}_m(\mathbb{F}_2)$ on the $\mathbb{F}_2$-vector space spanned by $t_1, \dots, t_m$ extends to an action on P_m and Φ_m (e_m is fixed in this action). Let $T_m \leq \mathrm{GL}_m$ be the upper triangular subgroup. We want to consider the rings of invariants

$$\Delta_m = \Phi_m^{T_m} \; ; \qquad \Gamma_m = \Phi_m^{\mathrm{GL}_m} \; .$$

Δ_m and Γ_m can be described as follows. We set

$$V_{k+1} = \prod \Big(\sum_{i=1}^{k} \lambda_i t_i + t_{k+1}\Big) , \qquad \lambda_i = 0, 1 ; \tag{1.2}$$

$$v_{k+1} = \frac{V_{k+1}}{e_k} \tag{1.3}$$

and define the elements $Q_{m,j} \in P_m$ inductively with the formulae

$$Q_{m,j} = Q_{m-1,j} Q_{m-1,0} v_m + Q_{m-1,j-1}^2$$

subject to the conventions

$$Q_{m,j} = \begin{cases} 1 & \text{if } m = j \geq 0, \\ 0 & \text{if } j < 0 \text{ or } m < j \; . \end{cases}$$

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If $I = (i_1, \dots, i_m)$ is a multi-index, with $i_j \in \mathbb{Z}$, we set

$$-I = (-i_1, \dots, -i_m)$$

and write v^I for the monomial $v_1^{i_1} \dots v_m^{i_m}$ (and similarly we write Sq^I or $Sq^{(i_1, \dots, i_m)}$ for the monomial $Sq^{i_1} \dots Sq^{i_m}$, where the Sq^j's are the Steenrod squares). In particular, we have

$$\Delta_m = \mathbb{F}_2[v_1^{\pm 1}, \dots, v_m^{\pm 1}] \ ; \qquad \Gamma_m = \mathbb{F}_2[Q_{m,0}^{\pm 1}, Q_{m,1}, \dots, Q_{m,m-1}] \ .$$

We observe that

$$P_m \cong H^*(RP^\infty \times \cdots \times RP^\infty) \qquad (m\text{-copies}) \ .$$

Here and in the sequel H^* indicates the mod 2 reduced cohomology functor. Hence P_m is acted upon by the mod 2 Steenrod algebra $\mathscr{A}$ and such an action extends, in a unique way, to an action on Φ_m (see [7]). Φ_m is a graded object: the grading is obtained by assigning degree 1 to each of the variables $t_1, \dots, t_m$. Now we consider the iterated total squaring operation S_m, defined as

$$S_m \colon H^*(X) \longrightarrow \Phi_m \otimes H^*(X) \qquad (X \text{ a } CW\text{-complex})$$

$$x \longmapsto \sum_{i_j \geq 0} (t_1^{-i_1} Sq^{i_1}) \cdots (t_m^{-i_m} Sq^{i_m})(x) \ .$$

S_m can be constructed in a purely algebraic way (as in [3]) or geometrically (e.g., see [2]).

Remark 1.4. If X is a CW-spectrum, S_m can still be defined, but $\Phi_m \otimes H^*(X)$ should be regarded as a completed tensor product (as in [1, p. 441]). In fact, when X is a spectrum, $H^*(X)$ is a stable $\mathscr{A}$-module and $S_m(x)$ is, in general, an infinite sum.

In this paper we exhibit an explicit nice formula for $S_m(x)$ as an element of $\Delta_m \otimes H^*(X)$. We show that

$$S_m(x) = \sum_I v^{-I} \otimes Sq^I(x), \qquad I = (i_1, \dots, i_m) \ ; \qquad i_j \geq 0 \ .$$

Moreover we construct a sequence of maps

$$\omega_m \colon \mathscr{A}_* \longrightarrow \Delta_m \, , \qquad m \geq 1 \, ,$$

where $\mathscr{A}_*$ denotes the $\mathbb{F}_2$-dual of $\mathscr{A}$. This construction allows us to give an alternative proof of a normalized version of a result of Múi's [3, Theorem 1, p. 346]. In fact, we show that

$$S_m(x) = \sum_R \omega_m(\xi^R) \otimes \xi_*^R(x) \tag{1.5}$$

where the sum runs over the multi-indices $R = (r_1, \dots, r_k)$ such that $r_i \geq 0$ for each $i = 1, \dots, k$ and $k \leq m$, $\xi^R = \xi_1^{r_1} \cdots \xi_k^{r_k}$ is a monomial in $\mathscr{A}_*$ and ξ_*^R indicates the corresponding element in the Milnor basis $\mathscr{B}$ of $\mathscr{A}$. We then show that the coefficient $\omega_m(\xi^R)$ that appears in the RHS of (1.5) equals the monomial $Q_{m,0}^{-r_1 \cdots - r_k} Q_{m,1}^{r_1} \cdots Q_{m,k}^{r_k}$ and (1.5) becomes

$$S_m(x) = \sum_R Q_{m,0}^{-r_1 \cdots - r_k} Q_{m,1}^{r_1} \cdots Q_{m,k}^{r_k} \otimes \xi_*^R(x) \ .$$

This is the announced normalized version of Múi's theorem. In particular, the above formula expresses the properties of invariance of the operation S_m. For related results, see also [4].

2. A NICE FORMULA FOR $S_m(x)$

This section is devoted to the proof of the following proposition.

Proposition 2.1. *Let* $x \in H^*(X)$. *We have*

$$S_m(x) = \sum_I v^{-I} \otimes Sq^I(x), \qquad I = (i_1, \dots, i_m) \ ; \qquad i_j \geq 0 . \tag{2.2}$$

Proof. As $v_1 = t_1$, the statement is trivial for $m = 1$. We use induction on m. We will assume the statement true for $m < n$ ($n \geq 2$) and prove it for $m = n$. We have

$$S_n(x) = \Big(\sum_{i_1 \geq 0} t_1^{-i_1} Sq^{i_1}\Big)\Big(\sum_{i_2, \dots, i_n \geq 0} (t_2^{-i_2} Sq^{i_2}) \dots (t_n^{-i_n} Sq^{i_n})(x)\Big) . \tag{2.3}$$

Our inductive hypothesis tells us that

$$\begin{aligned} S_{n-1}(x) &= \sum_{i_j \geq 0} (t_1^{-i_2} Sq^{i_2}) \dots (t_n^{-i_n} Sq^{i_n})(x) \\ &= \sum_{i_j \geq 0} v_1^{-i_2} \dots v_{n-1}^{-i_n} \otimes Sq^{(i_2, \dots, i_n)}(x) \\ &= \sum_{i_j \geq 0} \prod_{k=1}^{n-1} \left(\frac{\prod_{\lambda_j = 0,1} \left(\sum_{j=1}^{k-1} \lambda_j t_j + t_k \right)}{\prod_{\substack{\mu_1, \dots, \mu_{k-1} = 0,1 \\ \sum \mu_i > 0}} \sum_{j=1}^{k-1} \mu_j t_j} \right)^{-i_{k+1}} \otimes Sq^{(i_2, \dots, i_n)}(x) . \end{aligned} \tag{2.4}$$

In the last step above we have simply substituted each v_h with its rational expression in the t_j's, using (1.1), (1.2), and (1.3). Therefore, using (2.3) and (2.4), we get

$$S_n(x) = S_1 \left(\sum_{\substack{i_h \geq 0 \\ 2 \leq h \leq n}} \prod_{k=1}^{n-1} \left(\frac{\prod_{\lambda_j = 0,1} \left(\sum_{j=2}^{k} \lambda_j t_j + t_{k+1} \right)}{\prod_{\substack{\mu_j = 0,1 \\ \sum \mu_i > 0}} \sum_{j=2}^{k} \mu_j t_j} \right)^{-i_{k+1}} \otimes Sq^{(i_2, \dots, i_n)}(x) \right) . \tag{2.5}$$

In the above formula we have applied our inductive hypothesis using the set of variables $\{t_2, \dots, t_n\}$ instead of $\{t_1, \dots, t_{n-1}\}$. Since S_1 is a ring homomorphism (as is well known and easy to prove using the Cartan formula) we get

$$\begin{aligned} S_n(x) &= \sum_{i_2,\dots,i_n\geq 0}\prod_{k=1}^{n-1}\left(\frac{\prod(\sum\lambda_j S_1(t_j)+S_1(t_{k+1}))}{\prod\sum\mu_j S_1(t_j)}\right)^{-i_{k+1}}\otimes S_1(Sq^{(i_2,\dots,i_n)}(x)) \\ &= \sum_{i_2,\dots,i_n\geq 0}\prod_{k=1}^{n-1}\left(\frac{\prod(\sum\lambda_j S_1(t_j)+S_1(t_{k+1}))}{\prod\sum\mu_j S_1(t_j)}\right)^{-i_{k+1}}\otimes \sum_{i_1\geq 0} t_1^{-i_1} Sq^{(i_1,\dots,i_n)}(x) \\ &= \sum_{i_1,\dots,i_n\geq 0} v_1^{-i_1}\prod_{k=1}^{n-1}\left(\frac{\prod(\sum\lambda_j S_1(t_j)+S_1(t_{k+1}))}{\prod\sum\mu_j S_1(t_j)}\right)^{-i_{k+1}}\otimes Sq^I(x)\,. \end{aligned}$$

Here the λ_h's and the μ_l's are as in (2.5), I stands for $(i_1, \dots, i_n)$ and we use again the fact that $v_1 = t_1$. Hence we only need to check that

$$v_{k+1} = \frac{\prod(\sum\lambda_j S_1(t_j)+S_1(t_{k+1}))}{\prod\sum\mu_j S_1(t_j)}\,.$$

As t_j is a one-dimensional class, we have

$$S_1(t_j) = t_j + t_1^{-1}t_j^2, \qquad j = 2, \dots, n \quad \text{(see [6, Lemma 2.7, p. 6])}\,.$$

Thus

$$\begin{aligned} \prod_{\lambda_j=0,1}\left(\sum_{j=2}^{k}\lambda_j S_1(t_j)+S_1(t_{k+1})\right) &= \prod\left(\sum\lambda_j(t_j+t_1^{-1}t_j^2)+t_{k+1}+t_1^{-1}t_{k+1}^2\right) \\ &= t_1^{-2^k}\cdot\prod\left(\sum\lambda_j(t_1t_j+t_j^2)+t_1t_{k+1}+t_{k+1}^2\right)\,. \end{aligned}$$

Similarly

$$\prod_{\mu_2,\dots,\mu_k=0,1}\ \sum_{j=2}^{k}\mu_j S_1(t_j) = t^{-2^k+1}\cdot\prod\sum\mu_j(t_1t_j+t_j^2)\,.$$

Therefore

$$\text{(2.6)} \qquad \frac{\prod(\sum\lambda_j S_1(t_j)+S_1(t_{k+1}))}{\prod\sum\mu_j S_1(t_j)} = \frac{\prod(\sum\lambda_j(t_1t_j+t_j^2)+t_1t_{k+1}+t_{k+1}^2)}{t_1\cdot\prod\sum\mu_j(t_1t_j+t_j^2)}\,.$$

If we write A (B respectively) for the numerator (the denominator respectively) of the RHS of (2.6) above, we want to check that

$$A = V_{k+1}\ ; \qquad B = e_k\ .$$

We have

$$\begin{aligned}
V_{k+1} &= \prod_{\lambda_j=0,1} (\lambda_1 t_1 + \cdots + \lambda_k t_k + t_{k+1}) \\
&= \prod_{\lambda_j=0,1} (t_1 + \lambda_2 t_2 + \cdots + \lambda_k t_k + t_{k+1}) \cdot \prod_{\lambda_j=0,1} (\lambda_2 t_2 + \cdots + \lambda_k t_k + t_{k+1}) \\
&= \prod_{\lambda_j=0,1} ((t_1 + \lambda_2 t_2 + \cdots + \lambda_k t_k + t_{k+1})(\lambda_2 t_2 + \cdots + \lambda_k t_k + t_{k+1})) \\
&= \prod_{\lambda_j=0,1} (\lambda_2 t_1 t_2 + \cdots + \lambda_k t_1 t_k + t_1 t_{k+1} + (\lambda_2 t_2 + \cdots + \lambda_k t_k + t_{k+1})^2) \\
&= \prod_{\lambda_j=0,1} (\lambda_2 t_1 t_2 + \cdots + \lambda_k t_1 t_k + t_1 t_{k+1} + \lambda_2 t_2^2 + \cdots + \lambda_k t_k^2 + t_{k+1}^2) \\
&\qquad\qquad\qquad\qquad (\text{as } \lambda^2 = \lambda \text{ for } \lambda = 0,\, 1) \\
&= \prod_{\lambda_j=0,1} (\lambda_2(t_1 t_2 + t_2^2) + \cdots + \lambda_k(t_1 t_k + t_k^2) + t_1 t_{k+1} + t_{k+1}^2) = A\,.
\end{aligned}$$

A similar argument shows that $B = e_k$.

3. An alternative proof of a result of Múi's

We recall that the dual of the mod 2 Steenrod algebra $\mathscr{A}$ is a graded polynomial algebra

$$\mathscr{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$$

with grading given by setting $\deg(\xi_i) = 2^i - 1$.

As usual, for each multi-index $R = (r_1, \ldots, r_k)$ with each $r_i \geq 0$, we will write ξ^R for the monomial $\xi_1^{r_1} \ldots \xi_k^{r_k}$. As it is well known, the elements of $\mathscr{A}$ dual to the monomials ξ^R with respect to the basis of admissible monomials form a basis $\mathscr{B}$, called the Milnor basis of $\mathscr{A}$. The element of $\mathscr{B}$ dual to ξ^R is indicated by ξ_*^R.

We can define a map, which is formally identical to the iterated total squaring operation,

$$S_m \colon \mathscr{A} \longrightarrow \Delta_m \otimes \mathscr{A} \subseteq \Phi_m \otimes \mathscr{A}\,,$$
$$\alpha \longmapsto \sum_I v^{-I} \otimes Sq^I \circ \alpha\,,$$

with the proviso that $\Delta_m \otimes \mathscr{A}$ and $\Phi_m \otimes \mathscr{A}$ should be thought of as completed tensor products (as in Remark 1.4), because $\mathscr{A}$ is stable as a graded $\mathscr{A}$-module and $S_m(\alpha)$ is, in general, an infinite sum.

Definition 3.1. *Let $\omega_m \colon \mathscr{A}_* \longrightarrow \Delta_m$ be defined as follows. Let $\xi \in \mathscr{A}_*$, i.e., $\xi \colon \mathscr{A} \to \mathbb{F}_2$ is an $\mathscr{A}$-map, where $\mathbb{F}_2$ has the trivial $\mathscr{A}$-action. We set*

$$\omega_m(\xi) = \big((\mathrm{id} \otimes \xi) \circ S_m\big)(1) \qquad (1 \in \mathscr{A})\,.$$

In other words, ω_m is defined by the following diagram

$$\mathscr{A} \xrightarrow{S_m} \Delta_m \otimes \mathscr{A} \xrightarrow{\mathrm{id} \otimes \xi} \Delta_m \otimes \mathbb{F}_2 \cong \Delta_m$$
$$1 \longmapsto \omega_m(\xi)\,.$$

As

$$S_m(1) = \sum_I v^{-I} \otimes Sq^I \quad \text{(an infinite sum)}$$

we have

$$\omega_m(\xi) = (\mathrm{id} \otimes \xi)\left(\sum_I v^{-I} \otimes Sq^I\right) = \sum_I v^{-I} \cdot \langle \xi , Sq^I \rangle$$

where $\langle \xi , Sq^I \rangle$ is the value of the map ξ on Sq^I.

Proposition 3.2. ω_m *is a ring homomorphism.*

Proof. This is a straightforward calculation.

Proposition 3.3.

$$\omega_m(\xi_k) = \sum_I v^{-I}$$

where the sum runs over the multi-indices I *of the form*

$$I = (0, \ldots, 0, 2^{k-1}, 0, \ldots, 0, 2^{k-2}, \ldots, 1, 0, \ldots, 0)\ , \tag{3.4}$$

that is, I *is the multi-index* $(2^{k-1}, 2^{k-2}, \ldots, 2, 1)$ *with* $m-k$ *zeros inserted somewhere.*

Proof. ξ_k is dual to $M_k = Sq^{2^{k-1}} Sq^{2^{k-2}} \ldots Sq^1$ and it is easy to check that M_k does not appear in the admissible expression of any other monomial in $\mathscr{A}$. Therefore $\langle \xi_k , Sq^I \rangle = 1$ if and only if $Sq^I = M_k$, i.e., if and only if I is of the form (3.4).

Proposition 3.5.

$$\omega_m(\xi_k) = Q_{m,0}^{-1} Q_{m,k} \in \Gamma_m \subseteq \Delta_m \quad \forall\, m \geq 1\ .$$

Proof. See [2, Proposition 1, p. 39].

In other words, $Q_{m,0}^{-1} Q_{m,k}$ is the sum of all monomials v^{-I} with I of the form (3.4). From Propositions 3.2, 3.3, and 3.5 we deduce the following statement.

Corollary 3.6.

$$\omega_m(\xi^R) = Q_{m,0}^{-r_1 \ldots -r_k} Q_{m,1}^{r_1} \cdots Q_{m,k}^{r_k} \qquad (R = (r_1, \ldots, r_k))\ .$$

In [3] Múi defines a *non-normalized* version of S_m, which he calls F_m. By *non-normalized* we mean that F_m does not preserve the degrees; in fact, if $x \in H^n(X)$, the degree of $F_m(x)$ is $2^m \cdot n$ while $S_m(x)$ has degree n. Múi proves the following result [3, p. 346].

Theorem 3.7.

$$F_m(x) = \sum_R Q_{m,0}^{n-r_1 \ldots -r_k} Q_{m,1}^{r_1} \cdots Q_{m,k}^{r_k} \otimes \xi_*^R(x)$$

$$(R = (r_1, \ldots, r_k)\ ,\ r_i \geq 0\ ,\ x \in H^n(X))\ .$$

Corollary 3.6 allows us to give a very simple alternative proof of a normalized version of Theorem 3.7 above, using S_m instead of F_m.

Theorem 3.8.

$$S_m(x) = \sum_R Q_{m,0}^{-r_1 \dots -r_k} Q_{m,1}^{r_1} \dots Q_{m,k}^{r_k} \otimes \xi_*^R(x)$$

$$(R = (r_1, \dots, r_k) \ , \ r_i \geq 0 \ , \ k \leq m) \ .$$

Proof. We know, from (2.2), that

$$S_m(x) = \sum_I v^{-I} \otimes Sq^I(x) \ .$$

But the Milnor elements ξ_*^R form a basis for $\mathscr{A}$, therefore we have an expression of the form

$$S_m(x) = \sum_R \alpha(\xi^R) \otimes \xi_*^R(x)$$

where the $\alpha(\xi^R)$'s are suitable elements of $\Gamma_m \subset \Delta_m$. More precisely, for each R, $\alpha(\xi^R)$ is the sum of all the monomials v^{-I} with I such that $\xi_R(Sq^I) = 1$, i.e.,

$$\begin{aligned} \alpha(\xi^R) &= \omega_m(\xi^R) \\ &= Q_{m,0}^{-r_1 \dots -r_k} Q_{m,1}^{r_1} \dots Q_{m,k}^{r_k} \quad \text{(by Corollary 3.6)} \ . \end{aligned}$$

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