PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 115, Number 4, AUGUST 1992

LOCALLY COMPLETE INTERSECTION MULTIPLE STRUCTURES ON SMOOTH ALGEBRAIC CURVES

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(Communicated by Louis J. Ratliff, Jr.)

ABSTRACT. The aim of this paper is to characterize the class of the locally complete intersection multiple structures on smooth curves contained in a smooth three-dimensional variety.

INTRODUCTION

Let C be a smooth (connected) curve contained in a smooth three-dimensional variety X. In [1] Banica and Forster described all the multiple structures on C, i.e., the locally Cohen Macaulay curves $\overline{C} \subset X$ such that scheme theoretically $\overline{C} \supset C$ and $|\overline{C}|$—the underlying space of $\overline{C} = |C|$. $\overline{C}$ is called a quasi-primitive multiple structure on C if for almost all points $x \in C \operatorname{em}\dim_x \overline{C} = 2$.

The aim of this note is to characterize the quasi-primitive multiple structures that are locally complete intersections (lci). In order to make this paper self-contained, we present first the above-quoted results of Banica and Forster, which we shall need in the formulation of our result.

In the sequel C will be a smooth curve of a three-dimensional smooth (algebraic) variety X.

Definition. A locally Cohen Macaulay (lCM) curve $\overline{C} \subset X$ is called a multiple structure on C if $\overline{C} \supset C$ scheme theoretically and $|\overline{C}| = |C|$.

Let I and J denote the ideal sheaves of C and $\overline{C}$ respectively. For any $i \geq 1$ we define J_i as the minimal ideal sheaf containing $J+I^i$, which defines a lCM curve of X. Since J_i is obtained by removing all the embedded components of $J+I^i$, we infer that both considered ideals are[A generically equal. We have $J_1 = I$ and $J_i = J$ for $i \geq t+1$ where $t+1$ is the least i such that $J \supset I^i$. Moreover $J_i \supset J_{i+1}$ for $i \geq 1$.

We claim that $J_i \cdot J_j \subset J_{i+j}$. This is of course true in all the points of C where $J_i = J + I^i$. So the ideal $J_i \cdot J_j + J_{i+j}/J_{i+j} \subset O_X/J_{i+j}$ has a zero-dimensional support. Since O_X/J_{i+j} is lCM, this ideal is zero and therefore

Received by the editors November 13, 1989 and, in revised form, May 26, 1990.
1991 *Mathematics Subject Classification*. Primary 14H99.
This research was supported by the funds of the Ministry of Education.

$J_i \cdot J_j \subset J_{i+j}$. In particular $IJ_i \subset J_{i+1}$. So for all $i \geq 1$, J_i/J_{i+1} is a sheaf of O_X/I-modules. Since O_X/J_{i+1} is lCM and C is smooth, the O_X/I-modules J_i/J_{i+1} are locally free.

The multiplication map $J_i \times J_j \to J_{i+j}$ induces a generically surjective map $E_i \otimes E_j \to E_{i+j}$ where $E_i = J_i/J_{i+1}$. In particular, one has morphisms $E_1^{\otimes i} \to E_i$, which are also generically surjective.

Definition. With the above notation, a multiple structure on $C \subset X$ is called quasi-primitive if $\operatorname{rank} E_1 = 1$. Note that this implies that $\operatorname{rank} E_i \leq 1$ for $i \geq 1$. Actually $\operatorname{rank} E_i = 1$ for $1 \leq i \leq t$ and $E_i = 0$ for $i > t$.

Now we can formulate our

Theorem. *Let $\overline{C}$ be a quasi-primitive multiple structure on $C \subset X$. Then $\overline{C}$ is lci (i.e., J is locally generated by 2 elements) if and only if the morphisms $E_i \otimes E_{t-i} \to E_t$ are isomorphisms for $1 \leq i \leq t-1$.*

Remark. If C is projective then $E_i \otimes E_{t-i} \to E_t$ is an isomorphism if and only if $\deg E_i + \deg E_{t-i} = \deg E_t$.

Proof. The problem is local. So we can suppose that $O = O_X$ "is" a ring of a sufficiently small affine open neighborhood of a point of C. Since C is smooth, the epimorphism $O/J \to O/I$ whose kernel is the nilpotent ideal I/J splits. Therefore O/J admits a certain O/I-module structure and $O/J \simeq O/I \oplus I/J$ as O/I-modules.

For every $i \geq 1$, J_i/J becomes an O/I-submodule of O/J. The exact sequences of O/I-modules $0 \to J_{i+1}/J \to J_i/J \to E_i \to 0$ split since E_i is free. Therefore $O/J \simeq O/I \oplus J/I \simeq O/I \oplus E_1 \oplus \cdots \oplus E_t$ as O/I-modules $(J_i = J$ for $i \geq t+1)$ $O/I \oplus E_1 \oplus \cdots \oplus E_t$ carries a multiplicative structure that corresponds to the multiplication on O/J. We have

$$E_i \cdot E_j \subset \bigoplus_{i+j \leq k \leq t} E_k \quad \text{since } J_i/J \cdot J_i/J \subset J_{i+j}/J \text{ and } E_i \subset J_i/J.$$

Moreover the composed maps

$$E_i \otimes E_j \to E_i \cdot E_j \to \bigoplus_{i+j \leq k \leq t} E_k \xrightarrow{\text{projection}} E_{i+j}$$

coincide with the previously defined morphisms $E_i \otimes E_j \to E_{i+j}$.

Suppose that J is lci. This implies that O/J is Gorenstein. Since O/J is a finite module extension of O/I, there exists $\pi' \in \operatorname{Hom}_{O/I}(O/J, E_t)$ such that the homomorphism $O/J \to \operatorname{Hom}_{O/I}(O/J, E_t)$ induced by the bilinear form $(\ ,\ ): O/J \times O/J \to E_t$ $(r, s) = \pi'(rs)$ is an isomorphism [2]. Note that E_t is a rank 1 free O/I-module.

Let $\pi: O/J \to O/I$ be the projection induced by the decomposition $O/J \simeq O/I \oplus E_1 \oplus \cdots \oplus E_t$. There exists $s \in O/J$ such that $\pi(r) = \pi'(sr)$ for all $r \in O/J$. It is easy to see that s is invertible. Therefore we can assume that $\pi' = \pi$.

Put $E_0 = O/I$ and let γ denote the isomorphism $\bigoplus_{0 \leq i \leq t} E_i \to \operatorname{Hom}(\bigoplus_{0 \leq i \leq t} E_i, E_t)$ induced by the bilinear form $(r, s) = \pi(r \cdot s)$. Let $\gamma_{ij} \in \operatorname{Hom}(E_i, \operatorname{Hom}(E_j, E_t))$ denote the (i, j)th entry of the corresponding matrix. The elements $\gamma_{i,t-i}$ $0 \leq i \leq t$ are on its second diagonal. The elements below the second diagonal are zero since $E_i \cdot E_j = 0$ if $i + j > t$. It follows that γ is an isomorphism if and only if $\gamma_{i,t-i}$ is an isomorphism for $0 \leq i \leq t$. So the morphisms $E_i \otimes E_{t-i} \to E_t$ are isomorphisms since they are induced by $\gamma_{i,t-i}$.

Suppose now that for $1 \leq i \leq t-1$ the maps $E_i \otimes E_{t-i} \to E_t$ are isomorphisms. Then γ is an isomorphism since the maps $\gamma_{0t}\colon E_0 \to \mathrm{Hom}(E_t, E_t)$ and $\gamma_{t,0}\colon E_t \to \mathrm{Hom}(E_0, E_t)$ are obviously isomorphisms. To prove that J is a lci it suffices to invoke that "in codimension 2 case Gorenstein implies complete intersection."

References

1. C. Banica and O. Forster, *Multiplicity structures on space curves*, The Lefschetz centennial conference, part I, Contemp. Math., vol. 58, Amer. Math. Soc., Providence, RI, 1986, pp. 47–64.
2. R. Hartshorne, *Residues and duality*, Lecture Notes in Math., vol. 20, Springer-Verlag, Berlin, Heidelberg, New York, 1966.

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