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A CHARACTERIZATION OF THE SPHERE IN TERMS OF SINGLE-LAYER POTENTIALS

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ABSTRACT. Let Ω be a bounded smooth domain in $\mathbb{R}^n$, and suppose the single-layer potential of $\partial\Omega$ coincides for $y \notin \overline{\Omega}$ with the function $c|y|^{-1}$ $(c > 0)$. Then $\partial\Omega$ is a sphere centered at the origin.

Throughout this paper we assume $\Omega \subset \mathbb{R}^3$ (the case $\mathbb{R}^n$ is similar) is a bounded domain, and the boundary $\partial\Omega$ is smooth enough that for

$$u(y) = \int_{\partial\Omega} \frac{d\sigma_x}{|x-y|} \qquad \forall y \in \mathbb{R}^n, \tag{1}$$

one has

$$4\pi = \frac{\partial u}{\partial n^-} - \frac{\partial u}{\partial n^+} \quad \text{on } \partial\Omega, \tag{2}$$

where $d\sigma_x$ denotes the surface measure, $+$ indicates the limit from the exterior and $-$ the limit from the interior (see [K, p. 164]), and n is the outward normal vector to $\partial\Omega$, which we assume to exist. If Ω has this property, then we say Ω is smooth.

Theorem. *Let Ω be a bounded smooth domain in $\mathbb{R}^3$, and suppose for some $c > 0$*

$$\int_{\partial\Omega} \frac{d\sigma_x}{|x-y|} = \frac{c}{|y|} \qquad \forall y \neq \overline{\Omega}. \tag{3}$$

Then, $\partial\Omega$ is a sphere centered at the origin.

Proof. Since the single-layer potential is continuous in $\mathbb{R}^3$ (see [K, p. 160]) (3) implies $0 \in \Omega$. Therefore we can take two balls B_1, B_2 both centered at the origin, such that B_1 is the largest ball in Ω and B_2 is the smallest ball containing Ω. Now the idea is to show that $\partial B_1 = \partial B_2 = \partial\Omega$. Now let $y^1 \in \partial B_1 \cap \partial\Omega$ and $y^2 \in \partial B_2 \cap \partial\Omega$, then $|y^2| \geq |y^1|$. Thus it suffices to prove $|y^2| \leq |y^1|$. Define u to be the function represented by (1). Then by the maximum principle (since u is harmonic in Ω) max u in Ω is attained on

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$\partial\Omega$, hence by (3) at y^1. Similarly, $\min u$ in Ω is attained at y^2; therefore,

$$\frac{\partial u}{\partial n^-}(y^1) \geq 0 \quad \text{and} \quad \frac{\partial u}{\partial n^-}(y^2) \leq 0.$$

Moreover, at y^1 and y^2 we have

$$\frac{\partial}{\partial n^+} = \frac{\partial}{\partial |y|},$$

which applied to (2) yields

$$4\pi = \frac{\partial u}{\partial n^-} - \frac{\partial u}{\partial n^+} = \frac{\partial u}{\partial n^-} - c\frac{\partial |y|^{-1}}{\partial |y|} \geq \frac{c}{|y|^2} \quad \text{at } y^1,$$

i.e., $|y^1| \geq \sqrt{c/4\pi}$. Similarly

$$4\pi = \frac{\partial u}{\partial n^-} - \frac{\partial u}{\partial n^+} = \frac{\partial u}{\partial n^-} - c\frac{\partial |y|^{-1}}{\partial |y|} \leq \frac{c}{|y|^2} \quad \text{at } y^2,$$

i.e., $|y^2| \leq \sqrt{c/4\pi}$. This completes the proof.

Remark. As far as the proof of the theorem goes, we do not need equation (2) to hold on the entire boundary of Ω, but just at the points on $\partial\Omega$ where u attains its maximum and minimum.

The theorem can very easily be generalized to the case in which Ω is a union of disjoint domains or in which the density is assumed to be radially increasing. These generalizations, in the context of the *volume* potential, are dealt with in [ASZ].

We state these as corollaries, omitting the simple proofs.

Corollary 1. *Let*

$$\int_\Omega \frac{d\sigma_x}{|x-y|} = \sum_{j=1}^{m} \frac{c_j}{|x^j - y|} \qquad \forall\, y \notin \overline{\Omega},$$

where $\Omega = \bigcup_1^m \Omega_j$, $\{\Omega_j\}$ *are pairwise disjoint smooth domains, and* $x^j \in \Omega_j$. *Then,* $\partial\Omega_j$ *is a sphere centered at* x^j, $\forall_j$.

Corollary 2. *Let* $f = f(|x|)$ *be a continuous increasing function defined on* $\mathbb{R}^3$, *and suppose*

$$\int_{\partial\Omega} \frac{f(|x|)d\sigma_x}{|x-y|} = \frac{c}{|y|} \qquad \forall y \notin \overline{\Omega}.$$

Then $\partial\Omega$ *is a sphere centered at the origin.*

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