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NONLINEAR p-LAPLACIAN PROBLEMS ON UNBOUNDED DOMAINS

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ABSTRACT. We consider the p-Laplacian problem

$$-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u)+b(x)|u|^{p-2}u=f(x,u),$$
$$x\in\Omega,\ u|_{\partial\Omega}=0,\ \lim_{|x|\to\infty}u=0,$$

where $1<p<n$, $\Omega\ (\subset R^n)$ is an exterior domain. Under certain conditions, we show the existence of solutions for this problem via critical point theory.

1. INTRODUCTION

This paper is devoted to the study of the p-Laplacian problems

$$(*)\qquad \begin{aligned} lu&=f(x,u)\quad \text{in }\Omega,\\ u|_{\partial\Omega}&=0,\qquad \lim_{|x|\to\infty}u=0,\end{aligned}$$

where Ω is a smooth exterior domain in R^n (i.e., Ω is the complement of a bounded domain with $C^{1,\delta}$ boundary, $0<\delta<1$), $lu=-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u)+b(x)|u|^{p-2}u$, $1<p<n$, $0<a_0\le a(x)\in L^\infty(\Omega)\cap C^\delta(\overline{\Omega})$, $0\le b(x)\in L^\infty(\Omega)\cap C(\Omega)$. The objective is to obtain sufficient conditions on f for $(*)$ to have positive solutions in the following three prototype cases:

$$f(x,u)=\begin{cases} g(x)u^\alpha, & p-1<\alpha<p^*-1; \quad (1)\\ h(x)u^\beta, & 0\le\beta<p-1; \quad (2)\\ g(x)u^\alpha+h(x)u^\beta, & 0\le\beta<p-1<\alpha<p^*-1, \quad (3)\end{cases}$$

where $p^*=np/(n-p)$ is the Sobolev critical exponent. When $p=2$, $(*)$ is the usual second order elliptic problem, and (1), (2), and (3) correspond to the superlinear, sublinear, and mixed sub-superlinear cases respectively.

Several studies have appeared. For the case of bounded domains, we mention the works of Azorero and Alonso [2], Egnell [5], Guedda and Veron [6], and references therein. As to unbounded domains, we recall the results of Bidaut-Veron [4], Li and Yan [8], and Ni and Serrin [11]. Ni and Serrin [11] studied the radial case $\operatorname{div}(|\nabla u|^{p-2}u)+u^\alpha=0$ in R^n and showed that this equation

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admits no positive radial ground state solutions if $0 < \alpha < p^* - 1$ and, conversely, it does admit one if $\alpha \geq p^* - 1$. Li and Yan [8] considered the eigenvalue problem $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(x, u, \lambda) = 0$, $x \in R^n$, with $f(x, u, \lambda) = g(x, u) - \lambda |u|^{p-2}u$ or $\lambda(g(x, u) - |u|^{p-2}u)$, assuming $\lim_{t\to 0} g(x, t)/t^{p-1} = 0$, and obtained a decaying solution for $\lambda = \lambda_0$. Bidaut-Veron [4] studied the behaviour of solutions of $(*)$.

No existence theory seems to have been found to date for nonradially symmetric p-Laplacian problems of type $(*)$ in the cases (1), (2), and (3). As the problem $(*)$ has a variational structure, we naturally apply critical point theory to it. We first set up some weighted spaces in which the solutions are to be sought and for which the norm $\|u\|_l = (\int_\Omega a(x)|\nabla u|^p + b(x)|u|^p)^{1/p}$ induced by the operator l is an equivalent norm. Then we employ Mountain Pass arguments to obtain the existence of solutions. To prove the decay of the solutions, we make use of the estimates of Serrin [13].

2. Problem $(*)$ in case (1)

In this section, we consider problem $(*)$ in the case (1); that is, f is of the form $g(x)t^\alpha$ with $p - 1 < \alpha < p^* - 1$. We choose the function space E as the completion of $C_0^\infty(\Omega)$ under the norm $\|u\| = (\int_\Omega |\nabla u|^p + \omega|u|^p)^{1/p}$, where $\omega(x) = \max\{b(x), 1/(1 + |x|)^p\}$. From the definition, it is clear that $E \sim W_0^{1,p}(\Omega)$ if $b(x) \geq b_0 > 0$. Moreover, E has the following three important properties:

(a) E can be embedded into $W_{\mathrm{loc}}^{1,p}(\Omega)$;

(b) Sobolev Inequality: $\|u\|_{p^*} \leq A\|\nabla u\|_p$ for all $u \in E$, where $p^* = np/(n-p)$ and

$$A = \frac{n-1}{nv_n^{1/n}} \frac{\Gamma(n/p - 1)}{\Gamma(n/p)},$$

the Sobolev embedding constant (see [10, p. 56]), and $v_n = \operatorname{vol}(B_1(0))$;

(c) The norm $\|u\|_l = (\int_\Omega a|\nabla u|^p + b|u|^p)^{1/p}$ induced by the operator l is an equivalent norm on E.

Indeed, (a) and (b) are obvious, while (c) follows from a Hardy-type inequality

$$\int_\Omega \frac{1}{(1 + |x|)^p}|u|^p \leq C \int_\Omega |\nabla u|^p$$

for all $\varphi \in C_0^\infty(\Omega)$. This inequality can be obtained by applying the Divergence Theorem and Hölder inequality to the integral

$$\begin{aligned}\int_\Omega \frac{1}{(1 + |x|)^p}|u|^p &= -\frac{1}{n}\int x \cdot \nabla\left(\frac{1}{(1 + |x|)^p}|u|^p\right)\\ &\leq \frac{p}{n}\int_\Omega \left(\frac{1}{(1 + |x|)^{p-1}}|u|^{p-1}\right)(|\nabla u|) + \frac{p}{n}\int_\Omega \frac{1}{(1 + |x|)^p}|u|^p.\end{aligned}$$

We assume that f satisfies the following conditions:

(i) $f \in C^0(\Omega \times R^+)$, $f(x, t) > 0$ in $\Omega_0 \times (0, \infty)$ for some nonempty open $\Omega_0 \subseteq \Omega$;

(ii) $|f(x, t)| \leq g(x)|t|^\alpha$, $p - 1 < \alpha < p^* - 1$, $(0 \not\equiv)g \in L^\infty \cap L^{p_0}(\Omega)$ where $p_0 = \frac{np}{np - (\alpha+1)(n-p)}$;

(iii) there exists $\mu > p$ such that $\mu F(x, t) \le t f(x, t)$, $(x, t) \in \Omega \times R^+$, where $F(x, t) = \int_0^t f(x, s)\,ds$.

Now we define two functionals $K(u)$ and $J(u)$ on E,

$$K(u) = \int_\Omega F(x, u)\,dx, \qquad J(u) = \frac{1}{p}\|u\|_l^p - K(u). \tag{4}$$

$K(u)$ and $J(u)$ are well defined by assumption (ii) and Sobolev's Inequality. The functional $K(u)$ has the following basic properties.

Lemma 1. *Under* (i), (ii),

(a) $K(u)$ *is weakly lower semicontinuous and differentiable on* E *with* $K'(u)(\varphi) = \int_\Omega f(u, x)\varphi\,dx$ *for* $\varphi \in E$.

(b) $K'(u)$ *is a continuous and compact map from* E *to* E^*, *the dual of* E.

Proof. We select $\Omega_k = \{x \in \Omega |\, |x| \le r_k\}$, where r_k could be a fixed number or a sequence.

(a) Let $u_j \to u$ weakly in E. Observe that

$$\begin{aligned} |K(u_j) - K(u)| \le \int_{\Omega_k} & |F(x, u_j) - F(x, u)| \\ & + C\|g\|_{L^{p_0}(\Omega\backslash\Omega_k)}(\|u_j\|_l^{\alpha+1} + \|u\|_l^{\alpha+1}). \end{aligned} \tag{5}$$

Since $\{u_j\}$ is bounded in E, $\{u_j|_{\Omega_k}\}$ is bounded in $W^{1,p}(\Omega_k)$ for fixed k. It follows from the compact embedding $W^{1,p}(\Omega_k) \hookrightarrow L^q(\Omega_k)$ for $1 \le q < p^*$ (see, e.g., [1, Theorem 6.2]), that there exists a subsequence of $\{u_j\}$ that converges to u in $L^q(\Omega_k)$, whence $u_j \to u$ in $L^q(\Omega_k)$. Further, $\int_{\Omega_1} F(x, u_k) \to \int_{\Omega_1} F(x, u)$, since $|F(x, t)| \le \frac{1}{\alpha+1} g(x)|t|^{\alpha+1}$ and $1 < \alpha + 1 < p^*$. Therefore, $K(u_j) \to K(u)$, since (5) and $g \in L^{p_0}(\Omega)$.

For differentiability of K, we show that given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, u) > 0$ such that

$$\left| \int_\Omega F(x, u + \varphi) - \int_\Omega F(x, u) - \int_\Omega f(x, u)\varphi \right| < \varepsilon \|\varphi\|_l$$

for all $\varphi \in E$ with $\|\varphi\|_l \le \delta$. Observe that $g \in L^{p_0}(\Omega)$ and

$$\begin{aligned} &\left| \int_{\Omega\backslash\Omega_k} F(x, u + \varphi) - F(x, u) - f(x, u)\varphi \right| \\ &\quad \le \int_{\Omega\backslash\Omega_k} g\{(|u| + |\varphi|)^\alpha |\varphi| + |u|^\alpha |\varphi|\} \\ &\quad \le C\|g\|_{L^{p_0}(\Omega\backslash\Omega_k)}(\|u\|_l^\alpha + \|\varphi\|_l^\alpha)\|\varphi\|_l < \frac{\varepsilon}{2}\|\varphi\|_l \end{aligned}$$

for sufficiently large r_k and $\|\varphi\|_l \le 1$. To estimate the integral on the bounded domain Ω_k and obtain

$$\left| \int_{\Omega_k} F(x, u + \varphi) - F(x, u) - f(x, u)\varphi \right| < \frac{\varepsilon}{2}\|\varphi\|_l,$$

we need only follow the arguments in Proposition B10 of [12].

(b) The continuity of $K'(u)$ follows from the estimate

$$\begin{aligned}\|K'(u_j) - K'(u)\|_{E^*} \le C\{&\|f(\cdot\,, u_j) - f(\cdot\,, u)\|_{L^{np/(n+p)}(\Omega_k)} \\ &+ \|g\|_{L^{p_0}(\Omega\setminus\Omega_k)}(\|u_j\|_l^\alpha + \|u\|_l^\alpha)\},\end{aligned} \tag{6}$$

$1 < \alpha np/(n+p) < p^*$, and arguments similar to the one above. To show the compactness, we employ the diagonal method. Let $\{u_j\}$ be a bounded sequence in E, and let $r_k \to \infty$ such that $\|g\|_{L^{p_0}(\Omega\setminus\Omega_k)} < \frac{1}{k}$. For each k, the compactness of the embedding $W^{1,p}(\Omega_k) \hookrightarrow L^q(\Omega_k)(1 \le q < p^*)$ and the boundedness of $\{u_j\}$ in $W^{1,p}(\Omega_k)$ imply that $\{u_j\}$ has a Cauchy subsequence $\{u_{jk}\}$ in $L^q(\Omega_k)$. Then $\{K'(u_{jj})\}$ is a Cauchy sequence in E^*. Indeed, for given $\varepsilon > 0$, choose k such that $\|g\|_{L^{p_0}(\Omega\setminus\Omega_k)} < \varepsilon$. On the other hand, $\{u_{jj}\}$ $(\subset \{u_{jk}\})$ is convergent in $L^q(\Omega_k)$ $(1 \le q < p^*)$, and $|f(x\,, t)|^{np/(n+p)} \le C|t|^{\alpha np/(n+p)}$ with $1 < \alpha np/(n+p) < p^*$, whence for $j\,, k$ sufficiently large, $\|f(\cdot\,, u_{jj}) - f(\cdot\,, u_{ii})\|_{L^{np/(n+p)}(\Omega_k)} < \varepsilon$. The compactness of K' follows immediately from (6) by replacing u_j with u_{jj} and u with u_{ii}.

The critical points u of J, i.e.,

$$J'(u)(\varphi) = \int_\Omega (a|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + b|u|^{p-2}u\varphi - f(x\,, u)\varphi)\,dx = 0 \tag{7}$$

for all $\varphi \in E$ are weak solutions of $lu = f(x\,, u)$.

Lemma 2. *Let u be a critical point of J.*

(a) *$u \in L^q(\Omega)$, $\frac{np}{n-p} \le q < \infty$. If $b(x) \ge b_0 > 0$, then $u \in L^q(\Omega)$, $p \le q < \infty$,*

(b) $\lim_{|x|\to\infty} u = 0$.

Proof. (a) Let $u^+(x) = \max\{u(x)\,, 0\}$, $u^-(x) = \max\{-u(x)\,, 0\}$. We show that statement (a) is true for $u^\pm$. Set $u_k(x) = \min\{u^\pm(x)\,, k\}$, $k = 1\,, 2\,, \cdots$. For any real $i \ge 1$, $(u_k)^i \in E$. Substituting $\varphi = (u_k)^i$ in (7) and using $-\|g\|_\infty|t|^\alpha \le f(x\,, t) \le \|g\|_\infty|t|^\alpha$, we obtain

$$i\int_\Omega (u_k)^{i-1}|\nabla u_k|^p \le \frac{\|g\|_\infty}{a_0}\int_\Omega (u^\pm)^{\alpha+i}.$$

By the fact that $(u_k)^{i-1}|\nabla u_k|^p = (\frac{p}{i+p-1})^p|\nabla(u_k)^{(i+p-1)/p}|^p$ and Sobolev's Inequality, we have

$$\left(\int_\Omega (u_k)^{\frac{n}{n-p}(i+p-1)}\right)^{\frac{n-p}{n}} \le C\int_\Omega (u^\pm)^{\alpha+i}, \tag{8}$$

where $C = C(n\,, i\,, p\,, \alpha\,, a_0\,, \|g\|_\infty)$. Setting

$$\sigma = p^* - 1 - \alpha, \qquad i = i_0 = 1 + \sigma,$$
$$q_0 = \frac{n}{n-p}(i_0 + p - 1) = \frac{n}{n-p}(p + \sigma),$$

and letting $k \to \infty$ in (8), we conclude that $u^\pm \in L^{q_0}(\Omega)$. Iterating this process gives

$$(\|u_k\|_{q_{j+1}})^{\frac{n-p}{n}} \le C\int_\Omega (u^\pm)^{q_j},$$

where $q_j = \frac{n}{n-p}(i_j + p - 1)$, $i_j = 1 + \sigma + \frac{n}{n-2}\sigma + \cdots + (\frac{n}{n-2})^j\sigma$, and where $C = C(n, i_j, p, \alpha, a_0, \|g\|_\infty)$, $j = 0, 1, \cdots$. Thus, $u^\pm \in L^q(\Omega)$, $\frac{np}{n-p} \le q < \infty$, and (a) follows. To show (b), we choose r_0 sufficiently large so that $B_2(x) \subseteq \Omega$ for all x such that $|x| \ge r_0$. Then by Theorem 1 of Serrin [13], for some $q > \frac{n}{p}$,

$$\|u\|_{L^\infty(B_1(x))} \le C\{\|u\|_{L^{p^*}(B_2(x))} + \|f(x, u)\|_{L^q(B_2(x))}\},$$

where $C = C(n, p, q)$. The decay of u follows.

Now we can employ the Mountain Pass Theorem to obtain a solution of $(*)$.

Theorem 1. *Under conditions* (i)–(iii), *the problem* $(*)$ *has a positive decaying solution* $u \in C^{1,\delta}(\overline{\Omega} \cap B_r(0))$ *for any* $r > 0$ *and some* $\delta = \delta(r) \in (0, 1)$.

Proof. Since we seek positive solutions, it is convenient to define $f(x, t) = 0$ for $t \le 0$. By condition (ii), for small $r > 0$, there exists a $c > 0$ such that

$$J(u) \ge \frac{1}{p}\|u\|_l^p - C\|g\|_{p_0}\|u\|_l^{\alpha+1} \ge c$$

for $u \in \partial B_r(0)$. Integrating and using conditions (i) and (iii), we see that $F(x, t) \ge a_1 t^\mu - a_2$ for $(x, t) \in \Omega_0 \times R^+$, some $a_1, a_2 > 0$. Let $\varphi \in C_0^\infty(\Omega_0)$ such that $\varphi(x) \ge 0$, $\not\equiv 0$, and let $s \in R^+$. For s large,

$$J(s\varphi) \le \frac{1}{p}s^p\|\varphi\|_l^p - s^\mu \int_{\Omega_0} a_1\varphi^\mu + a_2|\Omega_0| < 0.$$

Thus we obtain the existence of e with $J(e) < 0$. To see that the (PS) condition holds, suppose $\{u_i\} \subseteq E$ is such that $J(u_i) \le C$ and $J'(u_i)(\cdot) \to 0$. Note that the inequality

$$C \ge J(u_i) \ge \frac{1}{p}\|u_i\|_l^p - \frac{1}{\mu}\int_\Omega f(x, u_i)u_i \ge \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_i\|_l^p + \frac{1}{\mu}J'(u_i)(u_i)$$

yields the boundedness of $\{u_i\}$. Consequently, it follows from the compactness of K' that there exists a subsequence of $\{u_i\}$, say $\{u_i\}$ itself, such that $K'(u_i)$ is Cauchy in E^*. We claim that $\{u_i\}$ is a Cauchy sequence in E. Indeed, we have the inequality

$$(9)\quad |\xi - \eta|^p \le \begin{cases} (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi - \eta) & \text{if } p \ge 2, \\ [(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi - \eta)]^{\frac{p}{2}}[|\xi|^p + |\eta|^p]^{\frac{2-p}{2}} & \text{if } 1 < p < 2 \end{cases}$$

for $\xi, \eta \in R^n$ (see, e.g., [7, 14]). On the other hand, from (7) we obtain

$$\begin{aligned} &\int_\Omega a(|\nabla u_i|^{p-2}\nabla u_i - |\nabla u_j|^{p-2}\nabla u_j)\cdot(\nabla u_i - \nabla u_j) \\ &\qquad + b(|u_i|^{p-2}u_i - |u_j|^{p-2}u_j)\cdot(u_i - u_j) \\ &\qquad \le |J'(u_i)(u_i - u_j)| + |J'(u_j)(u_i - u_j)| + \left|\int_\Omega (f(x, u_i) - f(x, u_j))(u_i - u_j)\right| \\ (10)\quad & \\ &\qquad \le C\{\|J'(u_i)\|_{E^*} + \|J'(u_j)\|_{E^*} + \|K'(u_i) - K'(u_j)\|_{E^*}\}, \end{aligned}$$

where $C = C(n, p)$. From (9) and (10), it follows immediately that $\{u_i\}$ is Cauchy in E. Thus the (PS) condition holds. The Mountain Pass Theorem guarantees the existence of a nontrivial critical point of J, say u. By Lemma 2, u decays. Letting $\varphi = u^-$ in (7) implies that $u \ge 0$ in Ω. The positivity $u(x) > 0$ in Ω follows from the Harnack type inequality [15, Theorem 1.1]. Finally, the proof of [9, Theorem 1] implies that $u \in C^{1,\delta}(\overline{\Omega} \cap B_r(0))$, $r > 0$.

To see this, let r be so large that $\partial\Omega \subset B_r(0)$. Since $u|_{\partial\Omega} = 0$, the boundary regularity arguments of [9, Theorem 1] can be applied in a neighborhood of $\partial\Omega$ in $\overline{\Omega} \cap B_r(0)$, while the interior regularity arguments of [9, Theorem 1] can be applied in the rest part of $\overline{\Omega} \cap B_r(0)$.

3. Other problems

We employ the method of the previous section, with some modifications, to consider the problem $(*)$ in cases (2) and (3)

$$f(x, u) = \begin{cases} h(x)u^\beta, & 0 \le \beta < p-1, \\ g(x)u^\alpha + h(x)u^\beta, & 0 \le \beta < p-1 < \alpha < p^* - 1. \end{cases}$$

We introduce the following conditions:

(iv) $0 \le f(x, t) \le h(x)|t|^\beta$, $0 \le \beta < p-1$, $h \in L^\infty(\Omega) \cap L^{q_0}(\Omega)$, $q_0 = \frac{np}{np-(\beta+1)(n-p)}$;

(v) $f(x, t) \ge h_0(x)t^{\beta_0}$ as $t \to 0^+$, $0 \le \beta_0 \le \beta$, $h_0(x) \ge 0, \not\equiv 0$.

Theorem 2. *Under conditions* (i), (iv), *and* (iv), *problem* $(*)$ *has a positive decaying solution* $u \in C^{1,\delta}(\overline{\Omega} \cap B_r(0))$ *for any* $r > 0$ *and some* $\delta = \delta(r) \in (0, 1)$.

Proof. We assume $f(x, t) = f(x, 0)$ for $t \le 0$ and define the functionals $K(u)$ and $J(u)$ on E as before. By condition (iv), the functional J is weakly lower semicontinuous differentiable. Moreover, J is bounded below, since

$$J(u) \ge \tfrac{1}{p}\|u\|_l^p - C\|h\|_{q_0}\|u\|_l^{\beta+1}.$$

Thus J has a critical point u: $J(u) = \inf\{J(v) \mid v \in E\}$, which is a solution of $(*)$. We note that u must be nontrivial since

$$J(s\varphi) \le \frac{s^p}{p}\|\varphi\|_l^p - \frac{s^{\beta_0+1}}{\beta_0+1}\int_\Omega h_0(x)|\varphi|^{\beta_0+1} < 0 \tag{11}$$

for some $\varphi \in C_0^\infty(\Omega)$ and small $s > 0$. The arguments of the nonnegativity, regularity, and decay of u in Theorem 1 work here. Since $lu \ge 0$, the weak Harnack inequality [15, Theorem 1.2] yields $u(x) > 0$ in Ω.

Theorem 3. *Let* $f(x, t) = f_1(x, t) + f_2(x, t)$. *Suppose that* f_1 *satisfies* (i)–(iii) *and* f_2 *satisfies* (i), (iv), *and* (v). *Then the problem* $(*)$ *has two positive decaying solutions* $u_1, u_2 \in C^{1,\delta}(\overline{\Omega} \cap B_r(0))$ *for any* $r > 0$ *and some* $\delta = \delta(r) \in (0, 1)$ *provided*

$$2A^p\|g\|_{p_0}^{\frac{p-\beta-1}{\alpha-\beta}} \cdot \|h\|_{q_0}^{\frac{\alpha+1-p}{\alpha-\beta}}\left[\frac{1}{\alpha+1}\left(\frac{(\alpha+1)(p-\beta+1)}{(\beta+1)(\alpha+1-p)}\right)^{\frac{\alpha+1-p}{\alpha-\beta}} + \frac{1}{\beta+1}\left(\frac{(\beta+1)(\alpha+1-p)}{(\alpha+1)(p-\beta-1)}\right)^{\frac{p-\beta-1}{\alpha-\beta}}\right] < 1,$$

where $A = \frac{n-1}{nv_n^{1/n}}\frac{\Gamma(n/p-1)}{\Gamma(n/p)}$, $v_n = \mathrm{vol}(B_1(0))$.

Proof. Once again, we employ Mountain Pass arguments to obtain the first nontrivial critical point of J. Here we assume $f(x, t) = f(x, 0)$ for $t \le 0$. In this case,

$$J(u) = \frac{1}{p}\|u\|_l^p - \int_\Omega K(u), \qquad K(u) = \int_\Omega F_1(x, u) + \int_\Omega F_2(x, u),$$

where $F_1(x, u) = \int_0^u f_1(x, s)\,ds$, $F_2(x, u) = \int_0^u f_2(x, s)\,ds$. J is weakly lower semicontinuous and differentiable, while K' is compact. Observe that for some $(0 \le)\ \varphi \in C_0^\infty(\Omega_0)$,

$$J(s\varphi) \le \frac{1}{p} s^p \|\varphi\|_l^2 - s^\mu \int_\Omega a_1 \varphi^u + a_2|\Omega_0| < 0$$

for large $s > 0$ and that

$$\begin{aligned} J(u) &= \frac{1}{p}\|u\|_l^p - \int_\Omega F_1(x, u) - \int_\Omega F_2(x, u) \\ &= \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u\|_l^p + \frac{1}{\mu} J'(u)(u) + \int_\Omega \left(\frac{1}{\mu} f_1(x, u)u - F_1(x, u)\right) \\ &\quad + \int_\Omega \left(\frac{1}{\mu} f_2(x, u)u - F_2(x, u)\right) \\ &\ge \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u\|_l^p + \frac{1}{\mu} J'(u)(u) - C\left(1 + \frac{1}{\mu}\right)\|h\|_{q_0}\|u\|_l^{\beta+1}. \end{aligned}$$

It follows that any sequence $\{u_i\}$ such that $J(u_i) \le C$ and $J'(u_i) \to 0$ is bounded. Thus $\{u_i\}$ has a convergent subsequence by the compactness of K' and $J'(u_i) \to 0$. The (PS) condition now follows, but the step $J(u) \ge a$ for $u \in \partial B_r(0)$ no longer follows as before. However,

$$\begin{aligned} J(u)|_{\|u\|_l = r} &\ge \left.\left(\frac{1}{p}\|u\|_l^p - \frac{1}{\alpha+1} A^{\alpha+1}\|g\|_{p_0}\|u\|_l^{\alpha+1} - \frac{1}{\beta+1} A^{\beta+1}\|h\|_{q_0}\|u\|_l^{\beta+1}\right)\right|_{\|u\|_l = r} \\ &= \frac{1}{p} r^p \left(1 - \frac{p}{\alpha+1} A^{\alpha+1}\|g\|_{p_0} r^{\alpha+1-p} - \frac{p}{\beta+1} A^{\beta+1}\|h\|_{q_0} r^{\beta+1-p}\right) \\ &\equiv \frac{1}{p} r^p H(r). \end{aligned}$$

Elementary differentiation shows that $H(r)$ has an absolute maximum

$$r_0 = \frac{1}{A}\left[\frac{(\alpha + 1(p - 1 - \beta)\|h\|_{q_0}}{(\beta+1)(\alpha+1-p)\|g\|_{p_0}}\right]^{\frac{1}{\alpha-\beta}}.$$

By assumption, $H(r_0) > 0$. Hence $J(u)|_{\|u\|_l = r_0} > 0$. By the Mountain Pass Theorem, $J(\cdot)$ has a critical point u_1 with $J(u_1) > 0$. Observe $J(u)|_{\|u\|_l = r_0} > 0$ and

$$J(s\varphi) \le \frac{s^p}{p}\|\varphi\|_l^p + s^{\alpha+1} C\|g\|_{p_0}\|\varphi\|_l^{\alpha+1} - \frac{s^{\beta_0+1}}{\beta_0+1}\int_\Omega h_0(x)|\varphi|^{\beta_0+1} < 0$$

for some $\varphi \in C_0^\infty(\Omega)$ and small $s > 0$. It follows that $J(\cdot)$ attains its local minimum at some $u_2 \in B_{r_0}(0)$, i.e., $J(u_2) = \inf\{J(v) \mid v \in B_{r_0}(0)\} < 0$. As before, we have $u_1, u_2 \ge 0$. Since u_1, u_2 satisfy $lu \ge f_1(x, u)$, the positivity of u_1, u_2 follows from [15, Theorem 1.2]. A slightly modified proof of Lemma 2 shows the decay of u_1, u_2; in this case the estimate of the proof of Lemma

2(a) proceeds as follows: $i \geq 1$,

$$\begin{aligned}\int_\Omega f(x, u)u^i &= \int_\Omega f_1(x, u)u^i + \int_\Omega f_2(x, u)u^i \\ &\leq \|g\|_\infty \int_\Omega u^{\alpha+i} + \int_{0\leq u\leq 1} hu^{\beta+i} + \int_{1\leq u} hu^{\beta+i} \\ &\leq (\|g\|_\infty + \|h\|_\infty) \int_\Omega u^{\alpha+i} + \int_\Omega hu^{\beta+1} \\ &\leq (\|g\|_\infty + \|h\|_\infty) \int_\Omega u^{\alpha+i} + C\|h\|_{q_0}\|u\|_l^{\beta+1}.\end{aligned}$$

The rest is the same. This completes the proof.

Suppose that $g(x) = O(|x|^{-\nu})$, $h(x) = O(|x|^{-\gamma})$ at ∞. Then conditions (ii) and (iv) imply that

$$\nu > \frac{np - (\alpha+1)(n-p)}{p}, \qquad \gamma > \frac{np - (\beta+1)(n-p)}{p}.$$

If $b(x) \geq b_0 > 0$, the assumption $g \in L^{p_0}(\Omega)$ of (ii) could be replaced by $\|g\|_{L^1(B_1(x))} \to 0$ as $|x| \to \infty$. In this case, we need only apply the embedding theorem of [3, Theorem 2.3] in the proofs above, where the property that $\|g\|_{L^{p_0}(\Omega\setminus\Omega_k)}$ can be arbitrarily small is used.

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