ON THE EXTENSION OF MULTIMEASURES AND INTEGRATION WITH RESPECT TO A MULTIMEASURE

DIMITRIOS A. KANDILAKIS

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We present a Caratheodory-Hahn-Kluvanek-type theorem for multimeasures whose values are weakly compact and convex subsets of a Banach space. Set functions obtained by integrating a bounded measurable function with respect to a multimeasure are also studied.

1. PRELIMINARIES

Let Ω be a set, Σ a field of subsets of Ω , and X a real Banach space. The following notations will be in use:

$$P_{(b)fc}(X) = \{A \subseteq X : \text{nonempty, (bounded) closed, convex}\}.$$

$$P_{wkc}(X) = \{A \subseteq X : \text{nonempty, weakly compact, convex}\}.$$

A set function $M: \Sigma \rightarrow 2^X \setminus \{\emptyset\}$ will be strongly additive if for $\{A_n\}_{n \in \mathbb{N}} \subseteq \Sigma$ pairwise disjoint, the series $\sum_{n \geq 1} x_n$ converges for any $x_n \in M(A_n)$. The range $M(\Sigma)$ of M is the set $\bigcup_{A \in \Sigma} M(A)$. If Σ is a σ -field, a multimeasure M is a map $M: \Sigma \rightarrow P_{fc}(X)$ such that $M(\emptyset) = \{0\}$ and for $\{A_n\}_{n \in \mathbb{N}} \subseteq \Sigma$ pairwise disjoint we have

$$M\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \geq 1} M(A_n).$$

There are several ways to interpret the sum of the sets in the right-hand side. In this work we will be using the following two:

(1) M is a (strong) multimeasure if for any $x_n \in M(A_n)$ the series $\sum_{n \geq 1} x_n$ converges in the norm and

$$\sum_{n \geq 1} M(A_n) = \left\{ x \in X : x = \sum_{n \geq 1} x_n (\text{unconditionally convergent}), \right. \\ \left. x_n \in M(A_n), n \in \mathbb{N} \right\}.$$

Received by the editors June 6, 1990 and, in revised form, February 9, 1991.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 28B20.

Key words and phrases. Set valued function, measure, multimeasure, support function, selector, weak compactness, uniformly strongly additive.

(2) M is a weak multimeasure if for every $x^* \in X^*$, $A \rightarrow \sigma(x^*, M(A)) = \sup_{x \in M(A)}(x^*, x)$ is a signed measure on $\mathbb{R} \cup \{+\infty\}$.

The above two definitions are equivalent in the case of $P_{wkc}(X)$ -valued multimeasures [3, Proposition 3].

Let Σ be a σ -field and $M: \Sigma \rightarrow P_{fc}(X)$ a multimeasure. If $A \in \Sigma$, let π be the collection of all finite Σ -partitions of A . We define:

$$|M|(A) = \sup_{\pi} \sum_{A_n \in \pi} h(\{0\}, M(A_n))$$

where h denotes the Hausdorff distance.

If $|M|(\Omega) < \infty$, then we say that $M(\cdot)$ is of bounded variation. Note that $|M|(\cdot)$ is itself and $\mathbb{R}_+$ -valued measure.

The space of all X -valued countably additive vector measures on Σ will be denoted by $M(X)$. A measure $m \in M(X)$ is a selector of a multimeasure $M: \Sigma \rightarrow P_{fc}(X)$ if $m(A) \in M(A)$ for every $A \in \Sigma$. The set of selectors of M will be denoted by S_M . Recall that S_M is decomposable, i.e., for $A \in \Sigma$ and $(m_1, m_2) \in S_M \times S_M$ the measure $X_A m_1 + X_{A^c} m_2 \in S_M$ (here $X_A m(B) = m(A \cap B)$ for all $B \in \Sigma$).

If $\mu: \Sigma \rightarrow \mathbb{R}_+$ is a countably additive measure and $M: \Sigma \rightarrow P_{fc}(X)$ a set function then M will be called continuous with respect to μ ($M \ll \mu$) if $\lim_{\mu(E) \rightarrow 0} h(\{0\}, M(E)) = 0$.

2. EXTENDING A STRONGLY ADDITIVE SET FUNCTION ON A σ -FIELD

In this section we provide the conditions for which a set function defined on a field Σ_0 can be extended to a multimeasure on the σ -field generated by Σ_0 .

Before going into our main theorem we need some auxiliary results. For the following assume that X is a real Banach space.

Proposition 2.1. *Let Σ_0 be a field of subsets of a set Ω , $M: \Sigma_0 \rightarrow P_{fc}(X)$ a set function such that $M(\Sigma_0)$ is a relatively weakly compact subset of X , the application $A \rightarrow \sigma(x^*, M(A))$ is a finitely additive measure for every $x^* \in X^*$. Then M is strongly additive.*

Proof. Let $\{A_n\}_{n \geq 1} \subseteq \Sigma_0$ pairwise disjoint and $x_n \in M(A_n)$. Since $\sigma(x^*, M(\cdot))$ is finitely additive we have $\sigma(x^*, M(\bigcup_1^k A_n)) = \sum_1^k \sigma(x^*, M(A_n))$ for every $x^* \in X^*$, therefore $M(\bigcup_1^k A_n) = \sum_1^k M(A_n)$ which implies that $\sum_1^k x_n \in M(\bigcup_1^k A_n) \subseteq M(\Sigma_0)$ which is weakly relatively compact, therefore each subseries $\sum_{k \geq 1} x_{n_k}$ has a weakly convergent subseries. We will show that for every $x^* \in X^*$ the series $\sum_{n \geq 1} \sigma(x^*, x_n)$ converges. If not, then there exists $x_0^* \in X^*$ such that the series $\sum_{n \geq 1} \sigma(x_0^*, x_n)$ diverges. Therefore the series of the positive terms or the series of the negative terms of $\sum_{n \geq 1} \sigma(x_0^*, x_n)$ diverges, a contradiction, since each subseries of $\sum_{n \geq 1} x_n$ —and consequently each subseries of $\sum_{n \geq 1} \sigma(x_0^*, x_n)$ —has a convergent subseries.

The conclusion now follows from Theorem IV.10.1 of [2].

Lemma 2.2. *If Σ_0 is a field of subsets of Ω and $M: \Sigma_0 \rightarrow 2^X \setminus \{\emptyset\}$ is a strongly additive set function then*

(i) $\lim_{n \rightarrow +\infty} h(\{0\}, \sum_{k=n}^{\infty} M(A_k)) = 0$ for any $\{A_n\}_{n \geq 1} \subseteq \Sigma_0$ pairwise disjoint. If we also assume that the application $A \rightarrow \sigma(x^*, M(A))$ is a finitely additive measure for every $x^* \in X^*$ then

(ii) $M(\Sigma_0)$ is a bounded subset of X .

If in addition $\sigma(x^*, M(\cdot))$ is countably additive for every $x^* \in X^*$ then

(iii) the collection $\{\sigma(x^*, M(\cdot)): x^* \in X^*, \|x^*\| \leq 1\}$ is a uniformly strongly additive family of measures.

Proof. (i) If $h(\{0\}, \sum M(A_k))$ does not converge to 0 as $n \rightarrow \infty$ then there exist $\varepsilon > 0$ and a subsequence $\{n_i\}_{i \in \mathbb{N}}$ of $\mathbb{N}$, such that $h(\{0\}, \sum_{k \geq n_i} M(A_k)) \geq \varepsilon$. Therefore for each i there exists $m_i \in \mathbb{N}$ and $x_i \in X$ with $\|x_i\| \geq \varepsilon/2$ such that $x_i \in \sum_{k=n_i}^{m_i} M(A_k)$ (here $\sum_{n \geq 1} M_n = \{x = \sum_{n \geq 1} x_n: x_n \in M_n\}$ with $M_n \subseteq X$, $n \in \mathbb{N}$). But then the series $\sum_{i \geq 1} x_{n_i}$ diverges, a contradiction.

(ii) As in the proof of Proposition 2.1 we have that M is finitely additive. If $M(\Sigma_0)$ is not bounded then there exists $\{A_n\}_{n \geq 1} \subseteq \Sigma$ such that $h(\{0\}, M(A_1)) > 1$, $A_n \subseteq A_{n+1} \forall n \in \mathbb{N}$, and $h(\{0\}, M(A_{n+1})) > 2h(\{0\}, M(A_n))$.

Take $x_1 \in M(A_1)$ such that $\|x_1\| \geq 1$ and $y_n \in A_n$ such that $\|y_{n+1}\| > 2h(\{0\}, M(A_n))$ for $n \geq 1$. Then $y_{n+1} = y'_{n+1} + x_{n+1}$ where $x_{n+1} \in B_{n+1} = A_{n+1} \setminus A_n$ and $y'_{n+1} \in A_n$. Therefore $\|x_{n+1}\| \geq h(\{0\}, M(A_n)) \geq 1$, which implies that the series $\sum_{n \geq 1} x_n$ diverges while $\{B_n\}_{n \geq 1} \subseteq \Sigma$ are pairwise disjoint, $x_n \in B_n$, and M is strongly additive, a contradiction.

(iii)

$$\begin{aligned} \lim_{n \rightarrow +\infty} \sup_{\|x^*\| \leq 1} \left| \sum_{k=n}^{\infty} \sigma(x^*, M(A_k)) \right| &= \lim_{n \rightarrow +\infty} \sup_{\|x^*\| \leq 1} \left| \sigma(x^*, \sum_{k=n}^{\infty} M(A_k)) \right| \\ &= \lim_{n \rightarrow +\infty} h \left(\{0\}, \sum_{k=n}^{\infty} M(A_k) \right) = 0, \end{aligned}$$

which implies that the collection $\{\sigma(x^*, M(\cdot)): x^* \in X^*, \|x^*\| \leq 1\}$ is a strongly countably additive family of countably additive measures on Σ_0 .

Proposition 2.3. Let Σ_0 be a field of subsets of a Ω and $M: \Sigma_0 \rightarrow P_{fc}(X)$ a strongly additive set function such that the applications $A \rightarrow \sigma(x^*, M(A))$ are countably additive real-valued measures for every $x^* \in X^*$. Then there exists a nonnegative real-valued countably additive measure μ on Σ_0 such that $M \ll \mu$.

Proof. By Lemma 2.2(iii) the collection $\{\sigma(x^*, M(\cdot)): x^* \in X^*, \|x^*\| \leq 1\}$ is a uniformly countably additive family of real-valued measures on Σ_0 . By the Caratheodory extension theorem for each $x^* \in X^*$, $\|x^*\| \leq 1$, there exists a unique countably additive measure $\sigma_{x^*}(\cdot)$ which extends $\sigma(x^*, M(\cdot))$ on the σ -field Σ generated by Σ_0 . Lemma 1 on p. 26 of [1] guarantees that the family $\{\sigma_{x^*}(\cdot): x^* \in X^*, \|x^*\| \leq 1\}$ is uniformly countably additive and Theorem 4 on p. 11 of [1] provides us with a nonnegative real-valued countably additive measure μ on Σ such that $\lim_{\mu(E) \rightarrow 0} \sup_{\|x^*\| \leq 1} \sigma_{x^*}(E) = 0$ which implies that $M \ll \mu|_{\Sigma_0}$.

Proposition 2.4. Let Σ be a σ -field of subsets of a set Ω and $M: \Sigma \rightarrow P_{wkc}(X)$ a multimeasure. Then $M(\Sigma)$ is a relatively weakly compact subset of X .

Proof. Let $x^* \in X^*$. The application $A \rightarrow \sigma(x^*, M(A))$ is a signed countably additive measure on Σ . By the Hahn decomposition theorem there exist two disjoint sets A, B in Σ whose union is Ω such that A is positive and B is

negative with respect to $\sigma(x^*, M(\cdot))$. Clearly $\sigma(x^*, M(A)) \geq \sigma(x^*, M(C))$ for every $C \in \Sigma$. Since $M(A)$ is weakly compact there exists $x_0 \in M(A)$ such that $(x^*, x_0) = \sigma(x^*, M(A))$. But then $(x^*, x_0) = \sigma(x^*, M(\Sigma))$ and the conclusion follows from the James theorem.

Proposition 2.5. *Let Σ_0 be a field of subsets of Ω , Σ the σ -field generated by Σ_0 , and $M_0: \Sigma_0 \rightarrow P_{bfc}(X)$ a set function such that the applications $A \rightarrow \sigma(x^*, M_0(A))$, $A \in \Sigma_0$, $x^* \in X^*$, are countably additive real-valued measures on Σ_0 . If there exists a nonnegative real-valued countably additive measure μ on Σ such that $M_0 \ll \mu|_{\Sigma_0}$ then M can be extended to a weak multimeasure $M: \Sigma \rightarrow P_{bfc}(X)$.*

Proof. We supply Σ with the pseudometric d defined by $d(E_1, E_2) = \mu(E_1 \Delta E_2)$, $E_1, E_2 \in \Sigma$. The subspace Σ_0 is then dense in Σ with respect to d . Since $M \ll \mu|_{\Sigma_0}$ the measures $\sigma(x^*, M_0(\cdot)): \Sigma_0 \rightarrow \mathbb{R}$ are uniformly continuous on Σ_0 for every $x^* \in X^*$. So each of them has a uniformly continuous extension $\sigma_{x^*}(\cdot): \Sigma \rightarrow \mathbb{R}$. It is easily verified that $\sigma_{x^*}(\cdot)$ is a countably additive μ -continuous measure on Σ for every $x^* \in X^*$. Since the application $x^* \rightarrow \sigma(x^*, M(A))$ is weakly lower semicontinuous for every $A \in \Sigma_0$, Σ_0 is dense in Σ , and $\sigma_{x^*}(\cdot)$ are uniformly continuous for every $x^* \in X^*$ we easily deduce that $x^* \rightarrow \sigma_{x^*}(A)$ is also weakly lower semicontinuous for every $A \in \Sigma$. We now define $M: \Sigma \rightarrow P_{bfc}(X)$ by $M(A) = \{x \in X: (x^*, x) \leq \sigma_{x^*}(A) \text{ for every } x^* \in X^*\}$, $A \in \Sigma$. The remarks on p. 4.8 of [5] guarantee that M is a weak multimeasure. Obviously M extends M_0 to Σ .

We are now ready to state the basic result of this section.

Theorem 2.6. *Let Σ_0 be a field of subsets of a set Ω and Σ the σ -field generated by Σ_0 . If $M_0: \Sigma_0 \rightarrow P_{wkc}(X)$ is a set function such that the set $\{\sigma(x^*, M_0(\cdot)): x^* \in X^*, \|x^*\| \leq 1\}$ is a family of countably additive measures on Σ_0 , then the following are equivalent:*

- (i) M_0 can be extended to a multimeasure $M: \Sigma \rightarrow P_{wkc}(X)$.
- (ii) There exists a nonnegative real-valued countably additive measure μ on Σ_0 such that $M_0 \ll \mu$.
- (iii) M_0 is strongly additive.
- (iv) $M_0(\Sigma_0)$ is a relatively weakly compact subset of X .

Proof. We only have to show that (i) implies (iv)

Clearly $M_0(\Sigma_0) = M(\Sigma_0) \subseteq M(\Sigma)$. By Proposition 2.4 $M(\Sigma)$ is relatively weakly compact so $M_0(\Sigma)$ is also relatively weakly compact.

3. THE INTEGRAL OF A BOUNDED MEASURABLE FUNCTION WITH RESPECT TO A MULTIMEASURE

In this section we define the integral of a bounded measurable function with respect to a multimeasure and study its properties.

Assume that (Ω, Σ) is a measurable space, X a Banach space, and $G: \Sigma \rightarrow P_{wkc}(X)$ a multimeasure. Let $B(\Sigma)$ be the space of all functions $f: \Omega \rightarrow \mathbb{R}$ which are Σ -measurable and bounded.

Definition 3.1. The integral of $f \in B(\Sigma)$ with respect to G on $A \in \Sigma$ is

$$\int_A f dG = \left\{ \int_A f dm: m \in S_G \right\}.$$

Remark. By Theorem 1 of [3] we know that S_G is not empty and Lemma 3 on p. 95 of [4] guarantees that the integrals on the right-hand side are well defined.

Theorem 3.2. *If $G: \Sigma \rightarrow P_{wkc}(X)$ is a multimeasure of bounded variation and $f \in B(\Sigma)$ then the set function defined by $M(A) = \int_A f dG$ for $A \in \Sigma$ is a $P_{wkc}(X)$ -valued multimeasure of bounded variation.*

Proof. First we will show that $M(A) \in P_{wkc}(X)$ for every $A \in \Sigma$. So let $\{\int_A f dm_n\}_{n \in \mathbb{N}}$ be a sequence in $M(A)$ where $\{m_n\}_{n \in \mathbb{N}} \subseteq S_G$. Since S_G is compact for the topology of simple weak pointwise convergence (see [6, §4]) there exists a subsequence $\{m_{n_k}\}_{k \in \mathbb{N}}$ of $\{m_n\}_{n \in \mathbb{N}}$ which converges to $m \in S_G$. But then for $x^* \in X^*$ we have:

$$\begin{aligned} \lim_{k \rightarrow +\infty} \left(x^*, \int_A f dm_{n_k} \right) &= \lim_{k \rightarrow +\infty} \int_A f d(x^*, m_{n_k}) \\ &= \int_A f d(x^*, m) = \left(x^*, \int_A f dm \right) \end{aligned}$$

(see [2, Example IV.13.18, p. 340]) which implies that $\int_A f dm_{n_k} \rightarrow \int_A f dm$. Therefore $M(A)$ is a weakly compact subset of X .

Now let $\{A_n\}_{n \in \mathbb{N}} \subseteq \Sigma$ pairwise disjoint. Then for $x^* \in X^*$ we have

$$\begin{aligned} \sigma \left(x^*, M \left(\bigcup_1^k A_n \right) \right) &= \sup_{m \in S_G} \left(x^*, \sum_{n=1}^k \int_{A_n} f dm \right) \\ (1) \quad &\leq \sum_1^k \sup_{m \in S_G} \left(x^*, \int_{A_n} f dm \right) \\ &= \sum_1^k \sigma(x^*, M(A_n)). \end{aligned}$$

For $n = 1, 2, \dots, k$ there exists $m_n \in S_G$ such that $\sigma(x^*, M(A_n)) = (x^*, \int_{A_n} f dm_n)$, $m_n \in S_G$, because $M(A_n)$ is w -compact. Since S_G is decomposable the measure $m: \Omega \rightarrow X$ defined by $m = X_{A_1} m_1 + \dots + X_{A_{k-1}} m_{k-1} + X_{\Omega - \bigcup_{i=1}^{k-1} A_i} m_k$ is a selector of G , therefore

$$\left(x^*, \int_{\bigcup_1^k A_j} f dm \right) = \sum_1^k \left(x^*, \int_{A_j} f dm_j \right) = \sum_1^k \sigma(x^*, M(A_j))$$

so

$$(2) \quad \sigma \left(x^*, M \left(\bigcup_1^k A_i \right) \right) \geq \sum_{i=1}^k \sigma(x^*, M(A_i)).$$

By (1) and (2) above we get that

$$\sigma \left(x^*, M \left(\bigcup_1^k A_i \right) \right) \geq \sum_{i=1}^k \sigma(x^*, M(A_i)).$$

Since $\bigcup_{i=1}^{\infty} A_i = (\bigcup_{i=1}^k A_i) \cup (\bigcup_{i=k+1}^{\infty} A_i)$ we have

$$\begin{aligned} \left| \sigma \left(x^*, M \left(\bigcup_1^{\infty} A_i \right) \right) - \sigma \left(x^*, M \left(\bigcup_1^k A_i \right) \right) \right| &= \left| \sigma \left(x^*, M \left(\bigcup_{k+1}^{\infty} A_i \right) \right) \right| \\ &= \left| \sup_{m \in S_G} \left(x^*, \int_{\bigcup_{k+1}^{\infty} A_i} f dm \right) \right| \leq \|f\|_{\infty} \cdot v \left(\sigma(x^*, G(\cdot)) \bigcup_{k+1}^{\infty} A_i \right) \end{aligned}$$

which converges to 0 as $k \rightarrow +\infty$ since G is of bounded variation (here $v(\sigma(x^*, G(\cdot)), A)$ denotes the variation of $\sigma(x^*, G(\cdot))$ on $A \in \Sigma$). Therefore M is a multimeasure. We can show that M is of bounded variation by an easy computation.

We will now compute the support function of M . Let $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$.

Theorem 3.3. *If G is as above then for $A \in \Sigma$*

$$\sigma(x^*, M(A)) = \int_A f^+ d\sigma(x^*, G(\cdot)) - \int_A f^- d\sigma(-x^*, G(\cdot)).$$

Proof. First assume that f is a simple function: $f = \sum_1^k a_i X_{A_i} + \sum_1^{\lambda} b_j X_{B_j}$, where $a_i > 0$, $i = 1, 2, \dots, k$, $b_j < 0$, $j = 1, 2, \dots, \lambda$, and $A_i \cap B_j = \emptyset$ for $i = 1, \dots, k$, $j = 1, \dots, \lambda$. Then for $C = \bigcup_1^k A_i$ and $D = \bigcup_1^{\lambda} B_j$ we have

$$\begin{aligned} \sigma(x^*, M(A)) &= \sigma \left(x^*, \int_A f dG \right) \\ &= \sigma \left(x^*, \int_C f dG \right) + \sigma \left(x^*, \int_D f dG \right) \\ &= \sup_{m \in S_G} \left(x^*, \int_C f dm \right) + \sup_{m \in S_G} \left(x^*, \int_D f dm \right) \\ &= \sup_{m \in S_G} \left(x^*, \sum_1^{\lambda} a_i m(A_i \cup A) \right) \\ &\quad + \sup_{m \in S_G} \left(-x^*, \sum_1^{\lambda} (-b_j) m(B_j \cap A) \right) \\ &= \int_A f^+ d\sigma(x^*, G(\cdot)) - \int_A f^- d\sigma(-x^*, G(\cdot)) \end{aligned}$$

and the result follows from a density argument.

We will now study the relation between S_M and S_G . So let $m \in S_G$. Then measure $n: \Sigma \rightarrow X$ defined by $n(A) = \int_A f dm$ for $A \in \Sigma$ is clearly a selector of M . Conversely let $n \in S_M$ and for $A \in \Sigma$ define

$$F(A) = \left\{ m(A): m \in S_M \text{ and } n(A) = \int_A f dm \right\}.$$

Clearly $F(A)$ is not empty for every $A \in \Sigma$ and it is easy to show that $F: \Sigma \rightarrow P_{wkc}(X)$ is also a multimeasure. By Theorem 1 of [3] F has a selector m which clearly satisfies $n(A) = \int_A f dm$ for every $A \in \Sigma$.

So we have proved the following.

Theorem 3.4. *Let M , G , and f be as above. Then:*

- (i) *If $m \in S_G$ the measure n defined by $n(A) = \int_A f dm$ is a selector of M .*
- (ii) *If $n \in S_M$ then there exists $m \in S_G$ such that $n(A) = \int_A f dm$ for every $A \in \Sigma$.*

In order to study the extreme structure of S_M we need the following lemma.

Lemma 3.5. *If $f \in B(\Sigma)$, $f(w) \neq 0$ for every $w \in \Omega$, $m \in M(X)$, and $\int_A f dm = 0$ for every $A \in \Sigma$, then m is identically zero on Σ .*

Proof. Suppose that there exists $A \in \Sigma$ such that $m(A) \neq 0$. Then either $m\{w \in A: f(w) > 0\} \neq 0$ or $m\{w \in A: f(w) < 0\} \neq 0$. Without loss of generality assume $m\{w \in A: f(w) > 0\} \neq 0$. Let $B = \{w \in \Omega: f(w) > 0\}$.

For $n \in \mathbb{N}$ define $A_n = \{w \in B: f(w) > \frac{1}{n}\}$. Then $\bigcup_1^\infty A_n = B$. Since $m(B) > 0$ there exists $k \in \mathbb{N}$ such that $m(A_k) \neq 0$. Take $x^* \in X^*$ such that $(x^*, m(A_k)) \neq 0$. Let $(x^*, m(\cdot))^+$ and $(x^*, m(\cdot))^-$ be the Jordan decomposition of $(x^*, m(\cdot))$ and B_1, B_2 the corresponding Hahn decomposition of A_k . Then either $(x^*, m(B_1))^+ \neq 0$ or $(x^*, m(B_2))^- \neq 0$. Without loss of generality assume that $(x^*, m(B_1))^+ \neq 0$. Then

$$0 = \left(x^*, \int_{B_1} f dm \right) = \int_{B_1} f d(x^*, m(\cdot)) \geq \frac{1}{n} (x^*, m(B_1))^+ > 0$$

which is a contradiction. Therefore $m(A) = 0$ for every $A \in \Sigma$.

Remark. If $f(w) \neq 0$ for all $w \in \Omega$ then the multimeasure F defined above is actually single valued.

Let $\text{ext } A$ denote the extreme points of the set $A \subseteq X$.

Theorem 3.6. *Let G, M be as above and $f(w) \neq 0$ for every $w \in \Omega$. Then if $n \in \text{ext } S_M$ there exists unique $m \in S_G$ such that $n(A) = \int_A f dm$ for $A \in \Sigma$ and if $n \in \text{ext } S_M$ the measure n defined by $n(A) = \int_A f dm$ for $A \in \Sigma$ is an extreme point of S_M .*

Proof. Let $n \in S_M$. The existence and uniqueness of an $m \in S_G$ s.t. $n(A) = \int_A f dm$ for $A \in \Sigma$ is guaranteed by Theorem 3.4 and the above remark. For the second part let $M \in S_G$ and $n \in S_M$ defined by $n(A) = \int_A f dm$ for $A \in \Sigma$. Suppose that $n \in \text{ext } S_M$. Then there exist $n_1, n_2 \in S_M$, $m_1, m_2 \in S_G$, and $a \in \mathbb{R}$ such that $n_1(A) = \int_A f dm_1$, $n_2(A) = \int_A f dm_2$, and $n(A) = an_1(A) + (1-a)n_2(A)$ for $A \in \Sigma$. Therefore

$$\int_A f dm = a \int_A f dm_1 + (1-a) \int_A f dm_2 \Rightarrow \int_A f d(m - am_1 - (1-a)m_2) = 0$$

for every $A \in \Sigma$.

By Lemma 3.5 above we get $m = am_1 + (1-a)m_2$, a contradiction which implies that $n \in \text{ext } S_M$.

REFERENCES

1. I. Deistel and J. J. Uhl, *Vector measures*, Math. Surveys, vol. 15, Amer. Math. Soc., Providence, RI, 1977.
2. N. Dunford and J. Schwartz, *Linear operators*, Part I, Wiley Interscience, New York, 1988.

3. C. Godet-Thobie, *Some results about multimeasures and their selectors*, Measure Theory at Oberwolfach, 1979 (Kölzow, ed.), Lecture Notes in Math., vol. 794, Springer, Berlin and New York, 1979, pp. 112–116.
4. I. Kluvanek and G. Knowles, *Vector measures and control systems*, North-Holland, Amsterdam, 1975.
5. R. Pallu de Barriere, *Multimeasures a valeurs convexes fermées*, Colloque sur l'Integration Vectoriel et Multivoque (A. Coste, ed.) expose no. 4, Caen, 1975.
6. N. S. Papargiou, *On the properties of multimeasures with values in a Banach space*, Comment. Math. Univ. Carolin. (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THE AEGEAN, 83200 KARLOVASSI, SAMOS, GREECE