A NOTE ON THE TRACE FORMULAS FOR ALMOST COMMUTING OPERATORS

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ABSTRACT. A formula related to the cyclic cohomology for the trace of the product of commutators is established.

1. INTRODUCTION

Let X and Y be two subalgebras (or groups) in an algebra A satisfying the condition that

$$[x^1, y^1] \cdots [x^k, y^k] \in J \quad \text{for } x^j \in X, y^j \in Y,$$

where $[\cdot, \cdot]$ is the commutator, J is the trace ideal of A , and k is a fixed natural number. Let τ be the trace on J . Define functions

$$\phi_n(x^0, \dots, x^n; y^0, \dots, y^n) = \tau([x^0, y^0] \cdots [x^n, y^n])$$

for $n \geq k-1$ and $\psi_n(x^0, \dots, x^n; y^0, \dots, y^n) = \tau(x^0 y^0 [x^1, y^1] \cdots [x^n, y^n])$ for $n \geq k$.

The following are some notations in cycle cohomology [1]. Let C^n be the space of functions $f_n(x^0, \dots, x^n)$, t be the operation $t: C^n \rightarrow C^n$ for every n

$$(tf_n)(x^0, \dots, x^n) = (-1)^n f_n(x^n, x^0, \dots, x^{n-1}),$$

and $C_\lambda^n = \{f \in C^n; tf = f\}$. Let

$$Af_n = (1 + t + \cdots + t^n)f_n \quad \text{for } f_n \in C^n;$$

then $A: C^n \rightarrow C_\lambda^n$. Let b be the Hochschild boundary operation

$$(bf_n)(x^0, \dots, x^{n+1}) = (b'f_n)(x^0, \dots, x^{n+1}) + (-1)^{n+1} f(x^{n+1}x^0, x^1, \dots, x^n),$$

where

$$(b'f_n)(x^0, \dots, x^{n+1}) = \sum_{j=0}^n (-1)^j f_n(x^0, \dots, x^j x^{j+1}, \dots, x^n).$$

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Let p be the operation

$$pf = \sum_{j=0}^n (n-j)t^j f \quad \text{for } f \in C^n,$$

where $t^0 = 1$. Define $Z_\lambda^n = \{f \in C_\lambda^n : bf = 0\}$ and $B_\lambda^n = bC_\lambda^{n-1}$. Define $S = bpb'$. The restriction of $2\pi iS$ at Z_λ^n coincides with Connes's S operator (cf. [1]), and this operator S also coincides with the constant multiple of some operator in [2].

In this paper the tensor product spaces (for example, $C^n \otimes C^n$ or $C_\lambda^n \otimes C_\lambda^n$) of functions of several variables $x^j \in X$ and $y^j \in Y$ are considered. The operations $t_x, A_x, b_x, \dots$ or $t_y, A_y, b_y, \dots$ are the corresponding operations $t, A, b, \dots$ with respect to x or y respectively. It is easy to see that $t_x \phi_n = t_y^{-1} \phi_n$; therefore, $A_x \phi_n = A_y \phi_n$, and it is denoted by $A\phi_n$.

We prove the following.

Theorem. For $n \geq k$, there is a $\Theta_n \in C_\lambda^n \otimes C_\lambda^n$ such that

$$(1) \quad A\phi_{n+1} = b_x b_y \Theta_n + \hat{\phi}_{n+1},$$

where

$$(2) \quad \hat{\phi}_{2m+p} = (-1)^m \prod_{j=1}^{2m-1} (k+j)^{-1} S_x^m S_y^m \phi_p$$

or

$$(3) \quad \hat{\phi}_{p+2m} = (-1)^m \prod_{j=1}^{2m} (k+j)^{-1} S_x^m S_y^m A\phi_p$$

for $p = k-1, k$. Besides, the function Θ_n is expressed by $\psi_n, \dots, \psi_k, \phi_{n-1}, \dots$, and ϕ_{k-1} .

The explicit form of Θ_n is given in (4), (5), and (38). This theorem is a generalization and refinement of the previous work [3].

The method of the proof of the theorem in §2 is the same as in [3], but the part of the proof in §3 is completely new. In order to make this note readable, we will rewrite some formulas in [3].

2. BASIC LEMMA

Define

$$\xi_n(x^0, x^1, \dots, x^n; y^1, \dots, y^n) = \tau(x^0[x^1, y^1] \cdots [x^n, y^n])$$

and

$$\eta_n(x^1, \dots, x^n; y^0, y^1, \dots, y^n) = \tau(y^0[x^1, y^1] \cdots [x^n, y^n]).$$

Let q be the operation

$$qf = nAf/2 - ptf, \quad f \in C^n,$$

for $n > 0$ and $qf = 0$ for $f \in C^0$. Let r be the operation

$$rf = r_n(t)f, \quad f \in C^n,$$

for $n > 0$, where $r_n(\cdot)$ is a polynomial of degree $\leq n - 1$ satisfying

$$qf = (1 - t)rf \quad \text{for } f \in C^n.$$

Besides, $rf = 0$ for $f \in C^0$. It is easy to see that $r_x\phi_n = r_y\phi_n$, and it is denoted by $r\phi_n$.

Define $\Theta_{k-2} = \Theta_{k-1} = 0$. Define

$$(4) \quad \zeta_n = \psi_n - \frac{1}{n}b_x b_y r\phi_{n-1} - \frac{1}{n(n-1)^2}S_x S_y \Theta_{n-2}$$

and

$$(5) \quad \Theta_n = \frac{1}{n+1}A_x A_y \zeta_n$$

for $n \geq k$. By the way, here the function Θ_n is the corresponding Θ_n in [3] multiplied by $n + 1$. Define

$$(6) \quad \hat{\xi}_k = \xi_k - \frac{1}{k}b_x q_y \phi_{k-1}, \quad \hat{\eta}_k = \eta_k - \frac{1}{k}q_x b_y \phi_{k-1}$$

and

$$(7) \quad \hat{\xi}_{n+1} = \frac{1}{(n+1)^2}S_x A_y \hat{\eta}_n, \quad \hat{\eta}_{n+1} = -\frac{1}{(n+1)^2}A_x S_y \hat{\xi}_n$$

for $n \geq k$ by mathematical induction.

Define

$$(8) \quad \hat{\zeta}_n = \frac{1}{(n+1)^2}A_x p_y b'_y \hat{\xi}_n + \frac{1}{n+1}p_x b'_x \hat{\eta}_n, \quad n = k, k+1, \dots$$

Later we will prove that

$$(9) \quad \hat{\zeta}_n = \frac{1}{(n+1)^2}p_x b'_x A_y \hat{\eta}_n + \frac{1}{n+1}p_y b'_y \hat{\xi}_n.$$

Define

$$(10) \quad \hat{\phi}_{n+1} = \frac{1}{n+1}S_x b_y A_y \hat{\eta}_n, \quad n = k, k+1, \dots$$

From (8), (9), $b_x \hat{\xi}_n = 0$, and $b_y \hat{\eta}_n = 0$, it is easy to see that

$$(11) \quad \hat{\phi}_{n+1} = \frac{1}{n+1}S_y b_x A_x \hat{\xi}_n.$$

By the way, the formulas for $\hat{\xi}_n$, $\hat{\eta}_n$, and $\hat{\phi}_n$ in [3] are incorrect for $n \geq 5$.

Lemma 1. For $n \geq k$,

$$(12) \quad A\phi_{n+1} = b_x b_y \Theta_n + \hat{\phi}_{n+1};$$

$$(13) \quad \zeta_n = \frac{1}{n+1} \left\{ \Theta_n + \frac{1}{n}(b_x p_y b'_y - p_x b'_x b_y) \Theta_{n-1} + q_y \phi_n \right\} + \hat{\xi}_n;$$

$$(14) \quad \xi_n = \frac{1}{n} \left(b_x \Theta_{n-1} - \frac{1}{n-1} S_x b_y \Theta_{n-2} + b_x q_y \phi_{n-1} \right) + \hat{\xi}_n;$$

$$(15) \quad \eta_n = \frac{1}{n} \left(-b_y \Theta_{n-1} - \frac{1}{n-1} b_x S_y \Theta_{n-2} + q_x b_y \phi_{n-1} \right) + \hat{\eta}_n.$$

Proof. Similar to Lemma 2 of [3], we have

$$(16) \quad (1 - t_x)\psi_n = b'_x\eta_n + \phi_n$$

and

$$(17) \quad (1 - t_y)\psi_n = b'_y\xi_n - t_y\phi_n$$

for $n \geq k$. From (16) and (17), we have

$$(18) \quad A_x\psi_n = (n+1)\psi_n - p_x(b'_x\eta_n + \phi_n)$$

and

$$(19) \quad A_y\psi_n = (n+1)\psi_n - p_y(b'_y\xi_n - t_y\phi_n)$$

for $n \geq k$. By (18) and (19), we get

$$(20) \quad A_x A_y \psi_n = (n+1)^2 \psi_n - (n+1)p_y b'_y \xi_n - p_x b'_x A_y \eta_n - (n+1)q_y \phi_n$$

$$(21) \quad = (n+1)^2 \psi_n - (n+1)p_x b'_x \eta_n - A_x p_y b'_y \xi_n - (n+1)q_y \phi_n,$$

since $t_y q_x \phi_n = -q_y \phi_n$.

On the other hand, from Lemma 1 of [3], we have

$$(22) \quad \xi_{n+1} = b_x \psi_n \quad \text{and} \quad \eta_{n+1} = -t_x b_y \psi_n$$

for $n \geq k$. From (22), we have

$$(23) \quad b_x A_x A_y \psi_n = (n+1)^2 \xi_{n+1} - S_x A_y \eta_n - (n+1)b_x q_y \phi_n$$

and

$$(24) \quad A_x b_y A_y \psi_n = -(n+1)^2 \eta_{n+1} - A_x S_y \xi_n + (n+1)b_y q_x \phi_n,$$

since $b_x \xi_n = 0$, $b_y \eta_n = 0$, and $t_x q_y \phi_n = -q_x \phi_n$. From Lemma 3 of [3], we have

$$(25) \quad A\phi_n = b_y A_y \xi_n, \quad A\phi_n = -b_x A_x \eta_n$$

for $n \geq k$. Applying A_y to both sides of (23) and A_x to both sides of (24), we get

$$b_x A_x A_y \psi_n = (n+1)A_y \xi_{n+1} - S_x A_y \eta_n$$

and

$$A_x b_y A_y \psi_n = -(n+1)A_x \eta_{n+1} - A_x S_y \xi_n,$$

since $b_x A\phi_n = b_y A\phi_n = 0$ (cf. (25)). Therefore,

$$(26) \quad b_x A_x b_y A_y \psi_n = (n+1)A\phi_{n+1} - S_x b_y A_y \eta_n$$

and

$$(27) \quad S_x b_y A_y \eta_n = b_x A_x S_y \xi_n.$$

From (6), it follows that (14) and (15) hold for $n = k$. Assume that (15) holds for $n \geq k$. Then by the calculation (cf. (26), (27), (28), and (29) in [3]), we have

$$(28) \quad \begin{aligned} S_x b_y A_y \eta_n &= -\frac{1}{n(n-1)^2} b_x A_x S_x b_y A_y S_y \Theta_{n-2} \\ &\quad - \frac{1}{n} b_x A_x b_x b_y A_y b_y r \phi_{n-1} + S_x b_y A_y \hat{\eta}_n. \end{aligned}$$

From (4), (5), (10), (26), and (28), we may prove that (12) holds. By (14), (15), and (27), we may prove (11). From (4), (5), (8), (14), (15), and (21), we have

$$\begin{aligned}
 & [(n+1)^2 - A_x A_y] \psi_n - (n+1) q_y \phi_n \\
 &= \frac{(n+1)}{n} [b_x p_y b'_y - p_x b'_x b_y] \Theta_{n-1} \\
 (29) \quad &+ \frac{1}{n(n-1)^2} [(n+1)^2 - A_x A_y] S_x S_y \Theta_{n-2} \\
 &+ \frac{1}{n} [(n+1)^2 - A_x A_y] b_x b_y r \phi_{n-1} + (n+1)^2 \hat{\xi}_n.
 \end{aligned}$$

This implies (13). Using (20) instead of (21), we may prove (9).

By (4), (13), and (22), we may prove that (7), (14), and (15) hold, where n is replaced by $n+1$, since $b_x \hat{\xi}_n = 0$ and $b_y \hat{\eta}_n = 0$. Therefore, Lemma 1 is proved by mathematical induction.

3. CALCULATION OF $\hat{\phi}_n$

In this section, we will calculate $\hat{\phi}_n$.

Lemma 2. *If $f \in C_\lambda^n$ then*

$$(30) \quad b A S f = (n+1) S b f.$$

Proof. By $b'(1-t) = (1-t)b$, we have

$$(31) \quad b' b f = -\frac{1}{(n+1)} (1-t) S f$$

(cf. [3, Lemma 4]). Therefore,

$$\begin{aligned}
 S b f &= b p b' b f = -\frac{1}{n+1} b p (1-t) S f \\
 &= \frac{1}{n+1} b (A - (n+3)) S f = \frac{1}{n+1} b A S f.
 \end{aligned}$$

Lemma 3. $b_y A_y \eta_k = -S_y \phi_{k-1}$ and $b_x A_x \xi_k = -S_x \phi_{k-1}$.

Proof. By direct calculation, it is easy to see that

$$(1-t_y) \eta_k = b'_y \phi_{k-1};$$

therefore,

$$t_y^m \eta_k = \eta_k - (1 + t_y + \cdots + t_y^{m-1}) b'_y \phi_{k-1}$$

for $m = 1, 2, \dots, k$. Thus,

$$A_y \eta_k = (k+1) \eta_k - p_y b'_y \phi_{k-1},$$

which proves that $b_y A_y \eta_k = -S_y \phi_{k-1}$. Similarly, we prove the other equality in Lemma 3.

Corollary 1. *If $m \geq 1$, then*

$$(32) \quad b_y (A_y S_y)^{m-1} A_y \hat{\eta}_k = - \prod_{j=1}^{m-1} (k+2j-1) S_y^{m-1} \left[S_y \phi_{k-1} + \frac{1}{k} q_x b_y A_y b_y \phi_{k-1} \right],$$

where $\prod_{j=1}^0(k+2j-1)$ is replaced by 1 and

$$(33) \quad S_x^m \phi_{k-1} \in C_\lambda^{k+2m-1} \otimes C^{k-1}.$$

Proof. By Lemma 3 and (6), equality (32) holds for $m = 1$. Assume that (32) holds for some $m \geq 1$. Then by Lemma 2 we have

$$\begin{aligned} b_y(A_y S_y)^m A_y \hat{\eta}_k &= (k+2m-1) S_y b_y(A_y S_y)^{m-1} A_y \hat{\eta}_k \\ &= - \prod_{j=1}^m (k+2j-1) S_y^m \left[S_y \phi_{k-1} + \frac{1}{k} q_x b_y A_y b_y \phi_{k-1} \right]. \end{aligned}$$

Thus, (32) holds for every $m \geq 1$ by mathematical induction. From (32) it is easy to see that

$$(34) \quad S_y^m \phi_{k-1} \in C^{k-1} \otimes C_\lambda^{k+2m-1}.$$

Changing $\hat{\eta}_k$ to $\hat{\xi}_k$, similar to (34), we may prove (33).

Corollary 2. For $m \geq 0$,

$$(35) \quad \begin{aligned} &\hat{\phi}_{k+2m+1} \\ &= (-1)^{1+m} \prod_{j=1}^{2m+1} (k+j)^{-1} \left[S_x^{m+1} S_y^{m+1} - \frac{1}{k} S_x^m b_x A_x b_x S_y^m b_y A_y b_y r \right] \phi_{k-1}. \end{aligned}$$

Proof. From (7), it is easy to see that

$$\hat{\eta}_{n+2} = - \frac{1}{(n+2)^2(n+1)^2} A_x S_x S_y A_y \hat{\eta}_n$$

for $n \geq k$. By (10), we have

$$(36) \quad \hat{\phi}_{k+2m+1} = (-1)^m (k+2m+1)^{-1} \prod_{j=1}^{2m} (k+j)^{-2} (S_x A_x)^m S_x b_y (A_y S_y)^m A_y \hat{\eta}_k.$$

Using (32) and (36), we get

$$(37) \quad \begin{aligned} \hat{\phi}_{k+2m+1} &= (-1)^{m+1} \prod_{j=1}^{2m+1} (k+j)^{-1} \prod_{j=1}^m (k+2j-1)^{-1} (S_x A_x)^m S_x \\ &\quad \times S_y^m \left[S_y \phi_{k-1} + \frac{1}{k} q_x b_y A_y b_y \phi_{k-1} \right]. \end{aligned}$$

By (33), we have $\prod_{j=1}^m (k+2j)^{-1} (S_x A_x)^m S_x \phi_{k-1} = S_x^{m+1} \phi_{k-1}$. From $b'(1-t) = (1-t)b$, we get

$$S_x q_x \phi_{k-1} = b_x p_x (1-t_x) b_x r \phi_{k-1} = -b_x A_x b_x r \phi_{k-1};$$

therefore, (37) implies (35). From (31), we have

$$\begin{aligned} S b A b f &= b p b' b A b f = -(n+2)^{-1} b p (1-t) S A b f \\ &= -(n+2)^{-1} b (n+4-A) S A b f = (n+2)^{-1} b A b p b' A b f \end{aligned}$$

for $f \in C^n$. Thus,

$$S^m b A b f = \prod_{j=1}^m (n+2j)^{-1} b A b (p b' A b)^m f$$

for $f \in C^n$. Hence,

$$\begin{aligned} & S_x^m b_x A_x b_x S_y^m b_y A_y b_y r \phi_{k-1} \\ &= \prod_{j=1}^m (k+2j)^{-2} b_x A_x b_x (p_x b'_x A_x b_x)^m b_y A_y b_y (p_y b'_y A_y b_y)^m r \phi_{k-1}. \end{aligned}$$

From (39), we only have to change Θ_{k+2m} in (1) to get

$$\begin{aligned} (38) \quad \Theta_{k+2m} &= \frac{1}{k+2m} A_x A_y \zeta_{k+2m} + (-1) \prod_{j=0}^{m+1} (k+2j)^{-2} \prod_{n=1}^m (k+2n-1)^{-2} \\ &\quad \times A_x b_x (p_x b'_x A_x b_x)^m A_y b_y (p_y b'_y A_y b_y)^m r \phi_{k-1}. \end{aligned}$$

Then (34) reduces to (2) for $p = k-1$. Since $(k-A)\phi_{k-1} = (1-t_x)(1-t_y) \times u(t_x)\phi_{k-1}$ for some u , we may prove (3) for $p = k-1$.

To prove (2) for $p = k$, instead of Lemma 3, we have to establish the following.

Lemma 4. $b_y A_y \hat{\eta}_{k+1} = -\frac{1}{k+1} S_y A \phi_k$ and $b_x A_x \hat{\xi}_{k+1} = -\frac{1}{k+1} S_x A \phi_k$.

Proof. By $b'(1-t) = (1-t)b$, (7), and (25), we have

$$\begin{aligned} (1-t_y)\hat{\eta}_{k+1} &= -\frac{1}{(k+1)^2} A_x (1-t_y) b_y p_y b'_y \hat{\xi}_k \\ &= -\frac{1}{(k+1)^2} A_x b'_y (k+1-A_y) b'_y \hat{\xi}_k = \frac{1}{(k+1)^2} A_x b'_y b_y A_y \hat{\xi}_k \\ &= \frac{1}{(k+1)^2} A_x b'_y A \phi_k - \frac{1}{k(k+1)^2} A_x b_x b'_y b_y A_y q_y \phi_{k-1} \\ &= \frac{1}{k+1} b'_y A \phi_k, \end{aligned}$$

since $A_y q_y \phi_{k-1} = 0$. Using the method in the proof of Lemma 3, we may prove Lemma 4.

By the same method in the proof of (2) and (3) for $p = k-1$, we may prove (2) and (3) for $p = k$.

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