BOUNDARIES OF THE SPECTRA IN $\mathcal{L}(X)$

WOO YOUNG LEE

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ABSTRACT. Suppose T is a bounded linear operator on a complex Banach space X . If T is "regular" with some finite dimensional intersection property and if 0 is in the boundary of the spectrum of T then 0 is an isolated point of it.

In [8], Putnam proved that if T is a bounded linear operator on a Hilbert space and if λ is an accumulation point of the boundary of the spectrum of T then either the range of $T - \lambda I$ is not closed or the dimension of the null space of $T - \lambda I$ is infinite.

In this note we generalize this result in the case of a Banach space.

Throughout this note suppose X and Y are complex Banach spaces, write $\mathcal{L}(X, Y)$ for the set of bounded linear operators from X to Y , and abbreviate $\mathcal{L}(X, X)$ to $\mathcal{L}(X)$. If $T \in \mathcal{L}(X)$ then we write $\alpha(T)$ and $\beta(T)$ for the $\dim T^{-1}(0)$ and $\text{codim } T(X)$; $\sigma(T)$ for the spectrum of T . If K is a compact subset of the complex plane $\mathbb{C}$, write ∂K and $\text{iso}(K)$, respectively, for the topological boundary points and the isolated points of K .

Recall that an operator $T \in \mathcal{L}(X, Y)$ is *regular* if there is $T' \in \mathcal{L}(Y, X)$ for which

$$(0.1) \quad T = TT'T.$$

If (0.1) holds then T' is called a *generalized inverse* for T . It is well known that T is regular if and only if $T(X)$ is closed and both $T^{-1}(0)$ and $T(X)$ are complemented (cf. [5, 6]). Recall, also, that $T \in \mathcal{L}(X, Y)$ is *upper semi-Fredholm* if $T(X)$ is closed and $\alpha(T) < \infty$ and is *lower semi-Fredholm* if $T(X)$ is closed and $\beta(T) < \infty$. If T is upper or lower semi-Fredholm it is called *semi-Fredholm*.

We begin with:

Lemma 1. *Suppose X is a Banach space and $X = Y \oplus Z$. If Y' is a closed subspace of X for which*

$$Y' \cong Y \quad \text{and} \quad Y' \cap Z \text{ is finite dimensional}$$

then $Y' + Z$ is closed in X .

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Proof. Suppose that $X = Y \oplus Z$, $Y' \cong Y$, and $P = P^2 \in \mathcal{L}(X)$ with $Z = P(X)$. If $Y' \cap Z$ is finite dimensional then we can find a closed subspace Y_0 for which

$$(1.1) \quad (Y' \cap Z) \oplus Y_0 = Y'.$$

Also, choosing subspaces W and W_0 of Y in such a way that

$$Y' \cap Z \cong W \quad \text{and} \quad W \oplus W_0 = Y,$$

then we have $W_0 \cong Y_0$ because $Y' \cong Y$; therefore $Z \oplus W_0 \cong Z \oplus Y_0$. We then can find a projection $Q = Q^2 \in \mathcal{L}(Y)$ with $W_0 = Q(Y)$. We now claim that

$$P + Q(I - P): X \rightarrow X$$

is a continuous projection on X : indeed, for each $x = y + z \in X$ with $y \in Y$ and $z \in Z$,

$$(P + Q(I - P))(x) = P(y + z) + Q(I - P)(y + z) = z + Q(y)$$

and

$$(P + Q(I - P))^2(x) = (P + Q(I - P))(z + Q(y)) = z + Q(Q(y)) = z + Q(y).$$

Therefore it follows at once that

$$(1.2) \quad (P + Q(I - P))(X) = Z \oplus W_0 \cong Z \oplus Y_0 = Z + Y'$$

is closed in X . The last equality of (1.2) comes from (1.1) and the observation that

$$\begin{aligned} x \in Z + Y' &\Rightarrow x = z + y_1 + y_2 \quad \text{with } z \in Z, \ y_1 \in Y' \cap Z, \text{ and } y_2 \in Y_0 \\ &\Rightarrow x \in Z + Y_0. \end{aligned}$$

Our main theorem is a generalization of the “punctured neighborhood theorem”:

Theorem 2. *Let X be a Banach space and let $T \in \mathcal{L}(X)$ be regular, with generalized inverse $T' \in \mathcal{L}(X)$. Suppose that $S \in \mathcal{L}(X)$ is invertible and commutes with T , and also satisfies that $I - ST'$ is invertible. If*

$$(2.1) \quad T(X) \cap T^{-1}(0) \quad \text{and} \quad (I - ST')T(X) \cap T^{-1}(0)$$

are finite dimensional then

$$(2.2) \quad T - S \text{ is upper semi-Fredholm}$$

and hence

$$(2.3) \quad 0 \in \partial\sigma(T) \Rightarrow 0 \in \text{iso } \sigma(T).$$

Proof. Suppose $T = TT'T$ is regular; thus $T'T(X)$ is the complementary subspace to $T^{-1}(0)$. Writing

$$M = (T - S)T'T(X) \cap T^{-1}(0),$$

then, by assumption, M is finite dimensional. Observe that, by assumptions,

$$(2.4) \quad (T - S)T'T(X) \cong T(X) \cong T'T(X) \quad \text{and} \quad ST^{-1}(0) = T^{-1}(0).$$

We now claim that

$T - S$ is upper semi-Fredholm.

Indeed, since $\dim M < \infty$, it follows from (2.4) that

$$\begin{aligned}(T - S)(X) &= (T - S)(T'T(X) \oplus T^{-1}(0)) \\ &= (T - S)T'T(X) + T^{-1}(0),\end{aligned}$$

which is closed in X , because the conditions of Lemma 1 are satisfied with $T^{-1}(0)$ in place of Z and $(T - S)T'T(X) = Y'$ and that for each $x = y + z$ with $y = T'T(w) \in T'T(X)$ and $z \in T^{-1}(0)$,

$$\begin{aligned}x \in (T - S)^{-1}(0) &\Rightarrow (T - S)(T'T(w)) - S(z) = 0 \\ &\Rightarrow (I - ST')T(w) \in M \text{ and } S(z) \in M \\ &\Rightarrow T(w) \in (I - ST')^{-1}(M) \text{ and } z \in S^{-1}(M) \\ &\Rightarrow x \in G + H\end{aligned}$$

with $G = T'(I - ST')^{-1}(M)$ and $H = S^{-1}(M)$,

where the second implication uses the second equality of (2.4) and $G + H$ must be finite dimensional; thus $\alpha(T - S)$ is finite. This proves (2.2).

Toward (2.3), apply the punctured neighborhood theorem to $T - S$ with $S = \mu I$ and $0 < |\mu| < \delta$; then $\alpha(T - \lambda I)$ and $\beta(T - \lambda I)$ are constant on a punctured neighborhood of 0 (see [3, Theorem V.1.8]). If $0 \in \partial\sigma(T)$ then it follows that for some θ with $0 < \theta < \delta$, $\alpha(T - \lambda I) = \beta(T - \lambda I) = 0$ for $0 < |\lambda| < \theta$; thus $T - \lambda I$ is invertible for $0 < |\lambda| < \theta$, giving (2.3).

We have been unable to decide whether or not the second condition of (2.1) can be dropped from Theorem 2. We conjecture that this is affirmative; we have not yet found a counterexample.

A semi-Fredholm operator T is *Fredholm* if both $\alpha(T)$ and $\beta(T)$ are finite. The *essential spectrum* of T , denoted $\sigma_e(T)$, is defined by

$$\sigma_e(T) = \{\lambda \in C: T - \lambda I \text{ is not Fredholm}\}.$$

If $T \in \mathcal{L}(X)$ is semi-Fredholm then the *index* of T is defined by

$$\text{index}(T) = \alpha(T) - \beta(T).$$

An operator $T \in \mathcal{L}(X)$ is *Weyl* if T is Fredholm and $\text{index}(T) = 0$. The *Weyl spectrum* of T , denoted $\sigma_w(T)$, is defined by

$$\sigma_w(T) = \{\lambda \in C: T - \lambda I \text{ is not Weyl}\}.$$

Both $\sigma_e(T)$ and $\sigma_w(T)$ are compact (cf. [1, 2, 5, 6]). If we define

$$P_n(T) = \{\lambda \in \sigma(T): T - \lambda I \text{ is semi-Fredholm and } \text{index}(T - \lambda I) = n\},$$

where $n \in \mathbb{Z} \cup \{\pm\infty\}$, then we can write

$$(2.5) \quad \sigma_w(T) = \sigma_e(T) \cup \left\{ \bigcup_{n \neq 0} P_n(T) \right\}.$$

For the essential spectrum and the Weyl spectrum, we can obtain the analogue of (2.3):

Theorem 3. *If X and T satisfy the hypotheses in Theorem 2, then there is implication*

$$(3.1) \quad 0 \in \partial \sigma_e(T) \Rightarrow 0 \in \text{iso } \sigma_e(T) \quad \text{and} \quad 0 \in \partial \sigma_w(T) \Rightarrow 0 \in \text{iso } \sigma_w(T).$$

Proof. By the same argument as Theorem 2, $(T - \lambda I)(X)$ is closed and both $\alpha(T - \lambda I)$ and $\beta(T - \lambda I)$ are constant on a punctured neighborhood of 0. Since $0 \in \partial \sigma_e(T)$, it follows that for some $\delta > 0$, both $\alpha(T - \lambda I)$ and $\beta(T - \lambda I)$ are finite for $0 < |\lambda| < \delta$. Thus we have $0 \in \text{iso } \sigma_e(T)$, giving the first implication of (3.1).

Toward the second implication, observe that for each $n \in \mathbb{Z} \cup \{\pm\infty\}$ with $n \neq 0$, $P_n(T)$ is a hole of $\sigma_e(T)$; thus by (2.5) the passage from $\sigma_e(T)$ to $\sigma_w(T)$ is the filling in certain of its holes; therefore we have $\partial \sigma_w(T) \subseteq \partial \sigma_e(T)$. This together with the first implication gives the result.

An operator $T \in \mathcal{L}(X, Y)$ is *decomposably regular* if there is $T' \in \mathcal{L}(Y, X)$ for which

$$T = TT'T \quad \text{and} \quad T' \text{ is invertible.}$$

It is well known that T is decomposably regular if and only if

$$(3.2) \quad T \text{ is regular} \quad \text{and} \quad T^{-1}(0) \cong Y/T(X).$$

Also if $T \in \mathcal{L}(X)$ then we have

$$(3.3) \quad \begin{aligned} &\{T \in \mathcal{L}(X): T \text{ decomposably regular}\} \\ &= \{T \in \mathcal{L}(X): T \text{ regular}\} \cap \text{cl}\{T \in \mathcal{L}(X): T \text{ invertible}\}. \end{aligned}$$

This follows from the result of Harte [4], wherein can be found a generalization.

As a special case of Theorem 2, we have:

Corollary 4. *Let X be a Banach space. If $T \in \mathcal{L}(X)$ is regular and if at least one of $\alpha(T)$ and $\beta(T)$ is finite then there is implication*

$$(4.1) \quad 0 \in \partial \sigma(T) \Rightarrow 0 \in \text{iso } \sigma(T).$$

Proof. Suppose $0 \in \partial \sigma(T)$ but T is regular; thus by (3.3) T is decomposably regular. If $\alpha(T) < \infty$ then trivially the condition (2.1) is satisfied; thus it follows from Theorem 2 that $0 \in \text{iso } \sigma(T)$.

If instead $\beta(T) < \infty$ then by (3.2) we also have $\alpha(T) < \infty$. This completes the proof.

We might remark that, remembering that if $T \in \mathcal{L}(X)$ then

$$T \text{ Fredholm} \Rightarrow T \text{ regular},$$

it clearly follows from Corollary 4 that

$$(4.2) \quad \partial \sigma(T) \setminus \sigma_e(T) \subseteq \text{iso } \sigma(T)$$

(see [5, (9.8.4.1)]); consequently (4.2) is a special case of Theorem 2.

If X is a Hilbert space and $T \in \mathcal{L}(X)$ then

$$T \text{ regular} \Leftrightarrow T(X) \text{ closed};$$

thus in this case Corollary 4 reduces to [2, Theorem XI.5.3].

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ADDED IN PROOF

We have recently shown that the second condition of (2.1) can be dropped from Theorem 2 (W. Y. Lee, *A generalization of the punctured neighborhood theorem*, Proc. Amer. Math. Soc. (to appear)).

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DEPARTMENT OF MATHEMATICS, SUNG KYUN KWAN UNIVERSITY, SUWON 440-746, KOREA