FIRST RETURN PATH DERIVATIVES

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ABSTRACT. A new type of path system is introduced. It is motivated by the Poincaré, first return, map of differentiable dynamics. Thus such systems are labeled first return path systems. It is shown that, though these are extremely thin paths, the systems possess interesting intersection properties that make the corresponding differentiation theory as rich as much thicker path systems.

1. INTRODUCTION

In this paper we introduce a new type of path system for the study of the differentiability structure of real functions. These systems are labeled as first return systems. The label and motivation comes from the Poincaré, first return, map of differentiable dynamics. If $F(x, t)$ is a differentiable flow in n -dimensional x space and p is a homoclinic fixed point of F , it is sometimes useful to construct an $(n - 1)$ -dimensional manifold M through p transverse to the flow. Then $P(x)$ is defined for x in M via the first return rule that $P(x) = F(x, t)$ for the minimum $t > 0$ such that $F(x, t)$ belongs to M . Then iteration of the function P can be used to study the behavior of the flow near p in place of F .

Further it is known for certain functions, such as $f(x) = (4x)(1 - x)$ on $[0, 1]$, that for almost any initial point z_0 the trajectory $z_{n+1} = f(z_n)$, $n = 0, 1, \dots$, will be dense in $[0, 1]$. This implies that for such a z_0 and any interval (a, b) there is a first time, or return, that the trajectory enters (a, b) . That is, we can define for z_0 , $r(a, b) = z_n$ where n is the minimal nonnegative index such that $a < z_n < b$. (Note. $n = 0$ is allowed.) It is tempting to replace the notation first return by first entry, but we have retained the slightly less accurate adjective to reflect the origins of the concept.

Now we will show how this idea can be adapted to be of use in differentiation theory. For this purpose it turns out that there are only two properties of a trajectory that are essential for our results. These are its order indexing and its denseness; that is, the fact that the trajectory is derived from a recursive formula is not essential. Therefore throughout this paper a trajectory will mean any sequence p_n , $n = 0, 1, \dots$, with the properties:

- (i) $p_0 = 0$, $p_1 = 1$,

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- (ii) $p_n \neq p_m$, $n \neq m$,
- (iii) $0 \leq p_n \leq 1$ for all n ,
- (iv) $\{p_n: n = 0, 1, \dots\}$ is dense in $[0, 1]$.

2. NOTATION AND DEFINITIONS

Our notation for a trajectory will simply be $\{p_n\}$.

For a given interval $(a, b) \subset [0, 1]$, $r(a, b)$ will be the first element of the trajectory in (a, b) .

For a given $k \geq 1$, π_k will represent the partition of the interval $[0, 1]$ generated by the initial segment $\{p_0, p_1, \dots, p_k\}$. Further the i th interval of that partition will be denoted as $\pi_{k,i}$.

In [1] the concept of path system was introduced and it was shown that many theorems about the differentiability of functions could be obtained from conditions on the thickness of the paths and how they intersect each other. Here our paths will be the thinnest possible in a bilateral sense and have some interesting nonstandard intersection properties. We will take many of our definitions from [1].

Definition 1. Let x belong to $[0, 1]$. A path leading to x is a set $P_x \subset [0, 1]$ such that $x \in P_x$ and x is a point of accumulation of P_x . The collection of paths, $\{P_x: x \in [0, 1]\}$, is called the path system $\mathcal{P}$.

Definition 2. Let $F: [0, 1] \rightarrow R$ and let $\mathcal{P}$ be any path system. If the

$$\lim_{\substack{y \rightarrow x \\ y \in P_x \setminus \{x\}}} \frac{F(y) - F(x)}{y - x} = f(x)$$

exists and is finite, then we say F is $\mathcal{P}$ -differentiable at x . If F is $\mathcal{P}$ -differentiable at every point x , then we say that F is $\mathcal{P}$ -differentiable and F is a $\mathcal{P}$ -primitive of f and f the $\mathcal{P}$ -derivative of F . We use the notation $f = F'_{\mathcal{P}}$.

We will now indicate how to use a trajectory to define a first return path system $\mathcal{R}$.

Definition 3. Let $\{p_n\}$ be a fixed trajectory. For $0 \leq y < 1$, R_y^+ = the right first return path to y is defined recursively and

$$y_1^+ = 1, \quad y_{k+1}^+ = r(y, y_k^+).$$

For $0 < y \leq 1$, R_y^- = the left first return path to y is defined similarly. For $0 < y < 1$, $R_y = R_y^- \cup R_y^+ \cup \{y\}$, and $R_0 = \{0\} \cup R_0^+$, $R_1 = R_1^- \cup \{1\}$.

In Definition 3 it may not be immediately obvious that for all y , R_y has y as a point of accumulation. Also there is an equivalent method of defining $\mathcal{R}$ based on π_k that will be used later in the paper. We incorporate these in the following lemma, which also lists some of the intersection properties of $\mathcal{R}$.

Lemma 1. For any given trajectory $\{p_n\}$, the first return path system has the following properties:

- (i) R_y^+ and R_y^- are monotone subsequences of p_n converging to y .
- (ii) For any open interval $(a, b) \subseteq [0, 1]$, $r(a, b)$ belongs to both R_a^+ and R_b^- .

(iii) For any partition π_k and partition interval $\pi_{k,i} = [c, d]$, c belongs to R_y^- for $c < y \leq d$ and d belongs to R_y^+ for $c \leq y < d$.

Further,

(iv) For any partition π_k and partition interval $\pi_{k,i} = [c, d]$, $R_y^- \cap [0, c] = R_z^- \cap [0, c]$ for $c < y, z \leq d$ and $R_y^+ \cap [d, 1] = R_z^+ \cap [d, 1]$ for $c \leq y, z < d$.

Proof. We will prove only (ii) and that only for a . The others follow by very similar arguments.

For (ii) and a consider $r(a, b) = p_N$. Then $N \geq 2$ and we know that $p_j \leq a$ or $p_j \geq b$ for $0 \leq j < N$. Let $A = \{p_j : 0 \leq j < N \text{ and } p_j \geq b\} \cap R_a^+$. Then $A \neq \emptyset$ because $1 = p_1$ belongs to R_x^+ for all $0 \leq x < 1$. Then $A = \{a_k^+, \dots, a_2^+, 1\}$ where $0 < k < N$ and $a_k^+ \geq b$. If $a < r(a, a_k^+) = p_m \geq b$, then $m < N$ must apply and $a_{k+1}^+ \in A$, which is a contradiction. Therefore $a < r(a, a_k^+) < b$ must hold and $r(a, a_k^+) = r(a, b)$.

To illustrate the above lemma and further elucidate the way the paths are generated consider a typical π_6

$$[0, p_3] \cap R_a^- = \{0, p_2, p_3\} = [0, p_3] \cap R_b^- \quad \text{and} \quad r(c, d) = p_3.$$

$$\text{Note } R_b^+ \cap [d, p_4] = \emptyset.$$

This explains that for given $0 \leq y < 1$, to find R_y^+ we can look at the sequence of partitions π_k and pick the partition interval $\pi_{k,i}$ of the form $[c_k, d_k)$ that contains y . Then R_y^+ will be the sequence d_k ; though we should note that we can have $d_k = d_{k+j}$ for $j > 0$; that is, we cannot say $d_k = y_k^+$.

Historically, if for a path-system $\mathcal{P}$, P_x consists of a single sequence converging to x for each x , then very little can be said about F or $F'_{\mathcal{P}}$.

For example, Sierpinski [6] showed that:

Theorem. Let h_n be any sequence of nonzero numbers converging to 0. Let for all x in R , $P_x = \{x + h_n, n = 0, 1, \dots\} \cup \{x\}$. Then for any finite function $f: R \rightarrow R$ there is a function F such that $F'_{\mathcal{P}} = f$ for all x .

For additional results of this type see [3, 5, 2]. Here we will show that, although any first return system is as thin as the above translated system, both F and $F'_{\mathcal{R}}$ are much better behaved.

Theorem 2. Let $\{p_n\}$ be any trajectory and $\mathcal{R}$ the first return path system generated by $\{p_n\}$.

Suppose $F: [0, 1] \rightarrow R$ is R -differentiable to f for all x . Then:

- (a) f is of Baire class 1;
- (b) F is Baire $\ast 1$, Darboux (see [4] for definition of Baire $\ast 1$);
- (c) f is Darboux;
- (d) For almost all x , the approximate derivative $F'_{\text{ap}}(x)$ exists and equals $f(x)$;
- (e) There is a dense open set U on which F is continuous and differentiable for almost all x .

Proof. The only hard part of this result is (a).

By [7] it is sufficient to show that there is a function $H(x, y)$ defined on $\{(x, y): 0 \leq x < y \leq 1\}$ with the property

$$\lim_{h \rightarrow 0^+} h(x - h, x) = \lim_{h \rightarrow 0^+} H(x, x + h) = f(x) \quad \text{for } 0 < x < 1.$$

Let $0 \leq x < y \leq 1$ be fixed. Consider $r(x, y) = p_N$, $N \geq 2$. For this N , let $[u, v]$ denote the partition interval of π_{N-1} that contains both x and y . We also label the interval found by this method $I(x, y)$.

We define $H(x, y) = (F(u) - F(v))/(u - v)$. We have to distinguish three cases in connection with the interval $I(x, y)$:

- (i) both x and y are in the trajectory segment $\{p_0, \dots, p_{N-1}\}$,
- (ii) neither x nor y are in the trajectory segment,
- (iii) only one of the two points x, y is in the trajectory segment.

In (i), $I(x, y) = [x, y]$. In (ii), $I(x, y) = [u, v]$ with $u < x, y < v$. In (iii), assuming without loss of generality that x belongs and y does not, then $I(x, y) = [x, v]$ with $y < v$.

By Lemma 1 we have for all these cases that $I(x, y)$ has both end points in R_x and R_y . In addition, let x be fixed and select any sequence $h_k \rightarrow 0^+$. Then $r(x, x + h_k) = p_{n_k} \rightarrow x$ and $n_k \rightarrow +\infty$. This, in turn, implies, in all three cases, applied to each $(x, x + h_k)$, that $I(x, x + h_k) = [u_k, v_k]$ has $u_k \leq x < v_k$ and $|u_k - v_k| \rightarrow 0$ and $u_k, v_k \in R_x$. So

$$H(x, x + h_k) = \frac{F(u_k) - F(v_k)}{u_k - v_k} \rightarrow f(x).$$

To obtain (b)–(e) we note that Lemma 1 part (ii) implies that any first return path system has the internal intersection property of [1]. This implies, by [1], that Theorem 11 of [4] holds and completes the proof.

A perusal of the previous proof of (a) reveals that its key feature is that a first return system, though thin, still satisfies an “outside” intersection property. That is, $I(x, y)$ contains $[x, y]$ and has both end points in $R_x \cap R_y$. This would lead to the conjecture that first return systems satisfy the external intersection property [1]. However, surprisingly, that *never* happens.

Definition 4. A path system $\mathcal{P}$ is said to have the external intersection property if there is a positive function $\delta(x)$ on $[0, 1]$ such that

$$P_x \cap P_y \cap (y, 2y - x) \neq \emptyset \quad \text{and} \quad P_x \cap P_y \cap (2x - y, x) \neq \emptyset,$$

whenever $0 < y - x < \min\{\delta(x), l\delta(y)\}$.

Proposition 3. Let $\{p_n\}$ be any trajectory and $\mathcal{R}$ the first return system generated by $\{p_n\}$. Then $\mathcal{R}$ does not have the external intersection property.

Proof. Assume the contrary. Let δ be the positive function of Definition 4. By the Baire category theorem there is a $\delta_0 > 0$ and an interval $(a, b) \subset (0, 1)$ such that for a dense subset S of (a, b) we have $\delta(x) > \delta_0$ for all x in S . We may assume $b - a < \delta_0$. Since the norm of the partitions π_k tends to 0, we can pick a k such that π_k contains two adjacent partitioning intervals entirely contained in (a, b) . Label these as I and J with $I = [c, d]$, $J = [d, e]$, and $a < c < d < e < b$. Then let $\varepsilon = \min((d - c)/3, (e - d)/3)$. Since S is dense in (a, b) we can pick two points x, y such that x, y belongs to S , $x < d < y$, and $|x - y| < \varepsilon$. (Note. $(y, y + (y - x)) \subset (d, e)$.)

We have $\min(\delta(x), \delta(y)) > \delta_0 > \varepsilon > |x - y|$, so we should have that $P_x \cap P_y \cap (y, y + (y - x)) \neq \emptyset$. Depending on the index of e in the trajectory, it might be that P_x and P_y share e , however, $P_x \cap (d, e) = \emptyset$, so we have a contradiction.

The above results point out how sensitive the process of first return differentiation is to the entries and indexing of the trajectory. Exactly how this affects the process will be the subject of a joint paper with Paul Humke and Michael Evans. However, we finish the paper with a result that shows a first return derivative is *essentially unique* and *independent* of the trajectory.

Proposition 4. *Let $\{p_n\}$ and $\{q_n\}$ be any two trajectories and $\mathcal{P}$ and $\mathcal{Q}$ the first return path systems they generate. Suppose a function F is first return differentiable to $\mathcal{F}'_{\mathcal{P}}$ and $\mathcal{F}'_{\mathcal{Q}}$ respectively. Then the set $\{x: F'_{\mathcal{P}}(x) \neq F'_{\mathcal{Q}}(x)\}$ is at most countable.*

Proof. As was mentioned in the proof of Theorem 2, the systems $\mathcal{P}$ and $\mathcal{Q}$ will have the internal intersection property; thus both $F'_{\mathcal{P}}$ and $F'_{\mathcal{Q}}$ are selective derivatives of F . The proof then follows from Theorem 9 of [4] or Theorem 7.8 of [1].

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