EXISTENCE OF SOLUTIONS TO $u'(t) = Au(t)$ FOR A WEAKLY CLOSED

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ABSTRACT. We give an existence theorem of solutions to the evolution equation $(d/dt)u(t) = Au(t)$ for a weakly closed operator A with some conditions on its resolvent in a reflexive Banach space.

INTRODUCTION

In this paper we consider the Cauchy problem

$$(CP_T; u_0) \begin{cases} (d/dt)u(t) = Au(t) & \text{for } t \in [0, T), \quad 0 < T < \infty, \\ u(0) = u_0 \end{cases}$$

for a given nonlinear operator A in a reflexive Banach space. In [2] we proved the existence of mild solutions to $(CP_T; u_0)$ for an operator A in a Hilbert space under the condition

$$(DD) \quad \langle Ax - A(x - \lambda Ax), \lambda Ax \rangle \leq 0 \\ \text{for } \lambda > 0, x \in D(A), \text{ and } x - \lambda Ax \in D(A)$$

and some other conditions. Our purpose is to give an existence theorem of solutions to $(CP_T; u_0)$ for a weakly closed operator A with a weaker condition than (DD). In §1 we state a theorem, and the proofs are given in §2.

1. NOTATION AND THEOREM

Let X be a reflexive Banach space with norm $\| \cdot \|$ and X^* its dual space. We shall denote by $\langle x, x^* \rangle$ the value of $x^* \in X^*$ at $x \in X$. Let A be a single-valued operator in X with domain $D(A)$ and range $R(A)$. We obtain the following

Theorem. Let $1 < p < \infty$ and $0 < T < \infty$. Let A be a single valued operator in X satisfying the following conditions:

- (A.1) for every $\lambda > 0$, $(1 - \lambda A)^{-1}$ is a single-valued operator;
- (A.2) $R(1 - \lambda A) \supset D(A)$ for all $\lambda > 0$;

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(A.3) for any $x \in D(A)$, there exists a positive number $K(x)$ such that

$$\sum_{k=1}^n \left(\frac{n}{T}\right)^{p-1} \left\| \left(1 - \frac{T}{n}A\right)^{-k} x - \left(1 - \frac{T}{n}A\right)^{-(k-1)} x \right\|^p \leq K(x)$$

for $n = 1, 2, 3, \dots$;

(A.4) A is a weakly closed operator, i.e., if $x_n \in D(A)$, $x_n \rightarrow x$ weakly and $Ax_n \rightarrow y$ weakly, then $x \in D(A)$ and $Ax = y$.

Under the above conditions, for given u_0 in $D(A)$, there exists a Hölder continuous function $u : [0, T] \rightarrow X$ satisfying the equation

$$u(t) = u_0 + \int_0^t Au(s) ds \quad \text{for all } t \in [0, T].$$

Remark. (DD), (A.1), and (A.2) imply (A.3) by taking $K(x) = T\|Ax\|^p$. This is easily checked, since (DD) yields the inequality

$$\|\lambda Ax\| \leq \|\lambda A(x - \lambda Ax)\|,$$

and by substituting $(1 - \lambda A)^{-k}x$ for x we have

$$\|(1 - \lambda A)^{-k}x - (1 - \lambda A)^{-(k-1)}x\| \leq \|(1 - \lambda A)^{-(k-1)}x - (1 - \lambda A)^{-(k-2)}x\|.$$

2. PROOF OF THEOREM

We begin with the following lemma.

Lemma 1. Let A be a single-valued operator in X satisfying conditions (A.1)–(A.3) of Theorem. Let $u_0 \in D(A)$, and put

$$u^n(t) = \begin{cases} u_k^n \equiv \left(1 - \frac{T}{n}A\right)^{-k} u_0 & \text{for } t \in \left(\frac{k-1}{n}T, \frac{k}{n}T\right], \\ u_0 & \text{for } t = 0. \end{cases}$$

Then $\{u^n\}$ has the following properties:

(i) $u^n(t) \in D(A)$ for all $t \in [0, \infty)$, and

$$u^n(t) = u_0 + \int_0^{\frac{k}{n}T} Au^n(s) ds \quad \text{for } t \in \left(\frac{k-1}{n}T, \frac{k}{n}T\right].$$

(ii) There exist a subsequence $\{u^{n_j}\}$ of $\{u^n\}$ and a function $u : [0, T] \rightarrow X$ such that

$$w\text{-}\lim_{j \rightarrow \infty} u^{n_j}(s) = u(s)$$

and

$$\|u(t) - u(s)\| \leq |t - s|^{\frac{1}{q}} (K(u_0))^{\frac{1}{p}} \quad \text{for all } t, s \in [0, T],$$

where $q = p/(p-1)$.

Proof. It is clear that $u^n(t) \in D(A)$ for all $t \in [0, \infty)$. If $t \in (\frac{k-1}{n}T, \frac{k}{n}T]$, then

$$\begin{aligned} u^n(t) &= u_k^n = u_0 + \sum_{j=1}^k (u_j^n - u_{j-1}^n) = u_0 + \sum_{j=1}^k \frac{T}{n} Au_j^n \\ &= u_0 + \sum_{j=1}^k \int_{\frac{j-1}{n}T}^{\frac{j}{n}T} Au^n(s) ds = u_0 + \int_0^{\frac{k}{n}T} Au^n(s) ds. \end{aligned}$$

By virtue of (A.3), we have

$$(1) \quad \begin{aligned} \int_0^T \|Au^n(s)\|^p ds &= \sum_{k=1}^n \int_{\frac{k-1}{n}T}^{\frac{k}{n}T} \left\| \frac{n}{T} (u_k^n - u_{k-1}^n) \right\|^p ds \\ &= \sum_{k=1}^n \left(\frac{n}{T} \right)^{p-1} \|u_k^n - u_{k-1}^n\|^p \leq K(u_0). \end{aligned}$$

Since $\{Au^n\}$ is a bounded sequence in reflexive Banach space $L^p([0, T]; X)$, there exist a subsequence $\{Au^{n_j}\}$ of $\{Au^n\}$ and a function v in $L^p([0, T]; X)$ such that

$$(2) \quad Au^{n_j} \rightarrow v \quad \text{weakly in } L^p([0, T]; X).$$

Therefore for any $w \in L^q([0, T]; X^*)$, we have

$$\int_0^T \langle Au^{n_j}(s) - v(s), w(s) \rangle ds \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

from which it follows that

$$(3) \quad \int_0^t \langle Au^{n_j}(s) - v(s), x^* \rangle ds \rightarrow 0 \quad \text{as } j \rightarrow \infty$$

for all $t \in [0, T]$ and $x^* \in X^*$. For $t \in (\frac{k-1}{n_j}T, \frac{k}{n_j}T]$, we have

$$(4) \quad u^{n_j}(t) = u_0 + \int_0^t Au^{n_j}(s) ds + \int_t^{\frac{k}{n_j}T} Au^{n_j}(s) ds.$$

Using Hölder's inequality and (1), we have

$$(5) \quad \begin{aligned} \left\| \int_t^{\frac{k}{n_j}T} Au^{n_j}(s) ds \right\| &\leq \int_t^{\frac{k}{n_j}T} \|Au^{n_j}(s)\| ds \leq \left(\frac{k}{n_j}T - t \right)^{\frac{1}{q}} (K(u_0))^{\frac{1}{p}} \\ &\leq \left(\frac{T}{n_j} \right)^{\frac{1}{q}} (K(u_0))^{\frac{1}{p}} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Hence, by (3) and (4),

$$(6) \quad w\text{-}\lim_{j \rightarrow \infty} u^{n_j}(t) = u_0 + \int_0^t v(s) ds \quad \text{for all } t \in [0, T].$$

Putting $u(t) = w\text{-}\lim_{j \rightarrow \infty} u^{n_j}(t)$, we have, by (2) and (1),

$$\|v\|_{L^p([0, T]; X)} \leq \liminf_{j \rightarrow \infty} \|Au^{n_j}\|_{L^p([0, T]; X)} \leq (K(u_0))^{\frac{1}{p}},$$

and hence, by (6),

$$\|u(t) - u(s)\| = \left\| \int_s^t v(s) ds \right\| \leq (t-s)^{\frac{1}{q}} (K(u_0))^{\frac{1}{p}}$$

for $0 \leq s < t \leq T$. $\square$

Lemma 2. Let A be a single-valued operator satisfying conditions (A.1)–(A.4) of Theorem, and let u be the weak limit function in (ii). Putting $S = \{s \in [0, T]; u(s) \in D(A)\}$, we get the following properties:

- (iii) S is a Lebesgue measurable set and the Lebesgue measure of S , denoted by $\mu(S)$, is $\mu(S) = T$.

- (iv) $Au \in L^p(S; X)$, i.e., $Au(\cdot)$ is strongly measurable in S , and $\int_S \|Au(s)\|^p ds < \infty$.

Proof. By Fatou's lemma, we have

$$(7) \quad \int_0^T \liminf_{j \rightarrow \infty} \|Au^{n_j}(s)\|^p ds \leq \liminf_{j \rightarrow \infty} \int_0^T \|Au^{n_j}(s)\|^p ds \leq K(u_0),$$

from which it follows that

$$(8) \quad \alpha(s) \equiv \liminf_{j \rightarrow \infty} \|Au^{n_j}(s)\| < \infty$$

for almost all $s \in [0, T]$. Fix $s \in [0, T]$ such that $\alpha(s) < \infty$. Then there exists a subsequence $\{n_{(j,k)}\}$ of $\{n_j\}$ such that

$$(9) \quad \lim_{k \rightarrow \infty} \|Au^{n_{(j,k)}}(s)\| = \alpha(s).$$

For any subsequence $\{Au^{n_{(j,k,l)}}(s)\}$ of $\{Au^{n_{(j,k)}}(s)\}$, there exist a subsequence $\{Au^{n_{(j,k,l,m)}}(s)\}$ of $\{Au^{n_{(j,k,l)}}(s)\}$ and $v(s) \in X$ such that

$$Au^{n_{(j,k,l,m)}}(s) \rightarrow v(s) \quad \text{weakly as } m \rightarrow \infty.$$

By virtue of (A.4) and (ii), we have

$$(10) \quad u(s) \in D(A) \quad \text{and} \quad w\text{-}\lim_{k \rightarrow \infty} Au^{n_{(j,k)}}(s) = Au(s).$$

Thus S is a Lebesgue measurable set, and $\mu(S) = T$. Moreover, we have, by (10), (9), and (8),

$$(11) \quad \|Au(s)\| \leq \alpha(s) = \liminf_{j \rightarrow \infty} \|Au^{n_j}(s)\|$$

for all $s \in [0, T]$ such that $\alpha(s) < \infty$. To prove (iv), we set

$$S_m = \{s \in S; \|Au(s)\| \leq m\}.$$

First we show that S_m is a closed set and $Au(\cdot)$ is weakly continuous on S_m . Assume that $s_n \in S_m$ and $s_n \rightarrow s$ as $n \rightarrow \infty$. For any subsequence $\{Au(s_{n_i})\}$ of $\{Au(s_n)\}$, there exist a subsequence $\{Au(s_{n_{(i,k)}})\}$ of $\{Au(s_{n_i})\}$ and $v(s) \in X$ such that

$$w\text{-}\lim_{k \rightarrow \infty} Au(s_{n_{(i,k)}}) = v(s).$$

By (ii) and (A.4), we have

$$u(s) \in D(A) \quad \text{and} \quad w\text{-}\lim_{n \rightarrow \infty} Au(s_n) = Au(s).$$

Thus

$$\|Au(s)\| \leq \liminf_{n \rightarrow \infty} \|Au(s_n)\| \leq m.$$

Therefore S_m is a closed set and $Au(\cdot)$ is weakly continuous on S_m . By standard argument, we can prove that $Au(\cdot)$ is strongly measurable in S_m (see [1, p. 73]). Since $S_m \uparrow S$ as $m \rightarrow \infty$, $Au(\cdot)$ is strongly measurable in S . Noting (8), we have, by (11) and (7),

$$(12) \quad \int_S \|Au(s)\|^p ds \leq K(u_0). \quad \square$$

Proof of Theorem. Setting

$$S_m^n = \{s \in S; \|Au^n(s)\| \leq m\}$$

and

$$v_m^n(s) = \begin{cases} Au^n(s) & \text{if } s \in S_m^n, \\ Au(s) & \text{if } s \in S \setminus S_m^n, \end{cases}$$

we have

$$(13) \quad \|v_m^n(s)\| \leq \max(m, \|Au(s)\|) \quad \text{for all } s \in S.$$

Moreover, we get, by (ii) and (A.4),

$$(14) \quad w\text{-}\lim_{j \rightarrow \infty} v_m^{n_j}(s) = Au(s) \quad \text{for all } s \in S \text{ and } m > 0.$$

By (i) and Lemma 2, for $t \in (\frac{k-1}{n_j}T, \frac{k}{n_j}T]$,

$$(15) \quad \begin{aligned} u^{n_j}(t) &= u_0 + \int_0^{\frac{k}{n_j}T} Au^{n_j}(s) ds \\ &= u_0 + \int_0^t v_m^{n_j}(s) ds + \int_0^t (Au^{n_j}(s) - v_m^{n_j}(s)) ds \\ &\quad + \int_t^{\frac{k}{n_j}T} Au^{n_j}(s) ds. \end{aligned}$$

By virtue of (13) and (14), using Lebesgue's dominated convergence theorem, we have

$$(16) \quad w\text{-}\lim_{j \rightarrow \infty} \int_0^t v_m^{n_j}(s) ds = \int_0^t Au(s) ds \quad \text{for } m > 0.$$

Noting that

$$K(u_0) \geq \int_0^T \|Au^n(s)\|^p ds \geq \int_{[0, T] \setminus S_m^n} m^p ds = m^p \mu([0, T] \setminus S_m^n) \quad \text{for } m > 0,$$

we have, by (1) and (12),

$$(17) \quad \begin{aligned} \left\| \int_0^t (Au^{n_j}(s) - v_m^{n_j}(s)) ds \right\| &\leq \int_{S \setminus S_m^{n_j}} \|Au^{n_j}(s) - Au(s)\| ds \\ &\leq \left(\int_{S \setminus S_m^{n_j}} 1^q ds \right)^{\frac{1}{q}} \left(\int_{S \setminus S_m^{n_j}} \|Au^{n_j}(s) - Au(s)\|^p ds \right)^{\frac{1}{p}} \\ &\leq (\mu([0, T] \setminus S_m^{n_j}))^{\frac{1}{q}} \left(\left(\int_0^T \|Au^{n_j}(s)\|^p ds \right)^{\frac{1}{p}} + \left(\int_S \|Au(s)\|^p ds \right)^{\frac{1}{p}} \right) \\ &\leq \left(\frac{K(u_0)}{m^p} \right)^{\frac{1}{q}} \cdot 2(K(u_0))^{\frac{1}{p}} \\ &= \frac{2}{m^{p-1}} K(u_0) \rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned}$$

In equation (15), first taking weak limit as $j \rightarrow \infty$ and next letting $m \rightarrow \infty$, by (i), (16), (17), and (5), we have

$$u(t) = u_0 + \int_0^t Au(s) ds \quad \text{for all } t \in [0, T]. \quad \square$$

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