ON THE REFLEXIVITY OF OPERATORS ON FUNCTION SPACES

K. SEDDIGHI AND B. YOUSEFI

(Communicated by Paul S. Muhly)

ABSTRACT. Let Ω be a bounded plane domain. Sufficient conditions are given so that an operator T in the Cowen-Douglas class $\mathcal{B}_n(\Omega)$ is reflexive. The operator M_z of multiplication by z on a Hilbert space of functions analytic on a finitely connected domain Ω is shown to be reflexive whenever $\sigma(M_z) = \overline{\Omega}$ is a spectral set.

1. INTRODUCTION

For a connected open subset Ω of the plane and n a positive integer, let $\mathcal{B}_n(\Omega)$ be the space introduced by Cowen and Douglas in [3]. This class of operators was further studied by Curto and Salinas [4]. In particular, they show that an operator T in $\mathcal{B}_n(\Omega)$ can be realized as the adjoint T_z^* of the operator of multiplication by $\bar{z}$ acting on a Hilbert space $\mathcal{H}_K$ of coanalytic functions on Ω having a generalized Bergman kernel K .

In this article we show that if Ω is a Carathéodory region and $T \in \mathcal{B}_n(\Omega)$ is such that $\sigma(T) = \overline{\Omega}$ is a spectral set then T is reflexive. As a result it follows that every contraction in $\mathcal{B}_n(\mathbf{D})$, where $\mathbf{D}$ is the open unit disk, is reflexive. Next we show that certain bilateral weighted shifts are reflexive. Finally, we consider a Hilbert space $\mathcal{H}$ of functions analytic on a finitely connected region Ω that contains the constants and admits multiplication by z as a bounded operator. We show that if $\sigma(M_z) = \overline{\Omega}$ is a spectral set then M_z is reflexive.

2. PRELIMINARIES

For a connected open subset Ω of the plane and n a positive integer, let $\mathcal{B}_n(\Omega)$ denote the operators T defined on the Hilbert space $\mathcal{H}$ that satisfy

- (a) $\Omega \subset \sigma(T)$,
- (b) $\text{ran}(T - \omega) = \mathcal{H}$ for ω in Ω ,
- (c) $\bigvee_{\omega \in \Omega} \ker(T - \omega) = \mathcal{H}$, and
- (d) $\dim \ker(T - \omega) = n$ for ω in Ω .

Received by the editors November 12, 1990.

1991 *Mathematics Subject Classification.* Primary 47B37; Secondary 47A25.

Key words and phrases. Reflexive, spectral set, Hilbert space of analytic functions, bilateral weighted shift.

Research partially supported by a grant (no. 69-SC-595-319) from Shiraz University Research Council.

The space $\mathcal{B}_n(\Omega)$ has been introduced by Cowen and Douglas [3]. Now let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Recall that if $A \in \mathcal{B}(\mathcal{H})$, then $\text{Lat}(A)$ is by definition the lattice of all invariant subspaces of A and $\text{Alg Lat}(A)$ is the algebra of all operators B in $\mathcal{B}(\mathcal{H})$ such that $\text{Lat}(A) \subset \text{Lat}(B)$. An operator A in $\mathcal{B}(\mathcal{H})$ is said to be *reflexive* if $\text{Alg Lat}(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $\mathcal{B}(\mathcal{H})$ that contains A and the identity I and is closed in the weak operator topology.

In [9] Sarason proved that normal operators are reflexive. It was shown by Deddens [5] that every isometry is reflexive. Olin and Thomson [8] showed that subnormal operators are reflexive. Bercovici, Foias, Langsam, and Pearcy [1] showed that (BCP)-operators are reflexive. The reflexive operators on a finite-dimensional space were characterized by Deddens and Fillmore [6].

It is shown in [4] that every operator in the class $\mathcal{B}_n(\Omega)$ is unitarily equivalent to the adjoint of the canonical model associated with a generalized Bergman kernel (g.B.k. for brevity) K .

Actually K is the reproducing kernel for a coanalytic functional Hilbert space $\mathcal{H}$ on which we define the operator $T_{\bar{z}}$ of multiplication by $\bar{z}$. The operator $T = T_{\bar{z}}^*$ acting on $\mathcal{H}$ is called the canonical model associated with K . We know that for every λ in Ω , $T - \lambda$ is onto, $\ker(T - \lambda) = \text{ran } K(\lambda, \cdot) = \{K(\lambda, \cdot)\zeta : \zeta \in \mathbb{C}^n\}$, and $\dim \ker(T - \lambda) = n$. See [4] for a detailed treatment of this subject.

The following characterization of the commutant $\{T\}'$ of T is given in Theorem 3.7 of [4], which is stated for the convenience of the reader.

Theorem 1. *If T is in $\mathcal{B}_n(\Omega)$ and the operator X commutes with T , then there exists an analytic function $\Phi: \Omega \rightarrow \mathcal{B}(\mathbb{C}^n)$ such that*

$$XK(\lambda, \cdot) = K(\lambda, \cdot)\Phi(\lambda) \quad (\text{all } \lambda \in \Omega)$$

and for every $f \in \mathcal{H}_K$, $X^*f(\cdot) = (\Phi(\cdot))^*f(\cdot)$.

3. CONDITIONS FOR THE REFLEXIVITY OF THE CANONICAL MODEL

In this section we give some sufficient conditions on the domain Ω so that the associated canonical model is reflexive. But first we prove the following

Lemma 1. *If T is in $\mathcal{B}_n(\Omega)$ then $\text{Alg Lat}(T) \subset \{T\}''$.*

Proof. Let $X \in \text{Alg Lat}(T)$. Each $x \neq 0$ in $\ker(T - z)$ spans a 1-dim invariant subspace of T and hence of X . Thus X restricted to $\ker(T - z)$ has the property that every nonzero vector is an eigenvector. So the restriction of X to $\ker(T - z)$ is a scalar multiple of the identity. That is, if $\{\gamma_1(z), \dots, \gamma_n(z)\}$ is a holomorphic spanning set for $\ker(T - z)$, then $X\gamma_i(z) = \psi(z)\gamma_i(z)$ for some complex number $\psi(z)$, for all i and for each z in Ω .

Fix z_0 in Ω . Since X is bounded, $X\gamma_i(z)$ is holomorphic, so locally $\psi(z)$ can be written as a quotient of holomorphic functions

$$\psi(z) = (\langle X\gamma_i(z), \gamma_i(z_0) \rangle) / (\langle \gamma_i(z), \gamma_i(z_0) \rangle).$$

Finally, $|\psi(z)|\|\gamma_i(z)\| = \|X\gamma_i(z)\| \leq \|X\| \cdot \|\gamma_i(z)\|$ implies $|\psi(z)| \leq \|X\|$, so ψ is in $H^\infty(\Omega)$. An application of Theorem 1 shows that $X \in \{T\}''$ and the proof of the lemma is complete. $\square$

In the next section we show that certain invertible bilateral weighted shifts with adjoint in $\mathcal{B}_1(\Omega)$ are reflexive. However, the case of a unilateral weighted shift can easily be taken care of.

Proposition 1. *If $T \in \mathcal{B}_1(\Omega)$ and T^* is an injective unilateral weighted shift, then T is reflexive.*

Proof. Theorem 1 of [12] states that $\{T^*\}' = W(T^*)$. Thus $\{T\}' = W(T)$. By Lemma 1 we have $\text{Alg Lat}(T) \subset \{T\}' \subset W(T)$. Therefore $\text{Alg Lat}(T) = W(T)$, which proves that T is reflexive. $\square$

We now make the following

Definition 1. An open connected subset U of the plane is called a *Carathéodory region* if its boundary equals the boundary of the unbounded component of $\mathbb{C} \setminus \bar{U}$.

It is easy to see that U is a Carathéodory region if and only if U is the interior of the polynomially convex hull of $\bar{U}$. In this case the Farrell-Rubel-Shields Theorem holds [7, Thorem 5.1, p. 151]. Let f be a bounded analytic function on U . Then there is a sequence $\{p_n\}$ of polynomials such that $\|p_n\|_U \leq C$ for a constant C and $p_n(z) \rightarrow f(z)$ for all $z \in U$.

Theorem 2. *If T is in $\mathcal{B}_n(\Omega)$, where Ω is a Carathéodory region such that $\sigma(T) = \bar{\Omega}$ is a spectral set for T , then T is reflexive.*

Proof. Let K be a g.B.k. on Ω and let $X \in \text{Alg Lat}(T)$. Then by Lemma 1 $XK(\lambda, \cdot) = \psi(\lambda)K(\lambda, \cdot)$ for some function $\psi \in H^\infty(\Omega)$. By the Farrell-Rubel-Shields Theorem, there is a uniformly bounded sequence $\{p_n\}$ of polynomials converging pointwise to ψ . Since $\sigma(T) = \bar{\Omega}$ is a spectral set we conclude that $\|p_n(T)\| \leq \|p_n\|_\Omega \leq C$ for some constant C . By passing to a subsequence we may assume that $p_n(T) \rightarrow A$ in the weak operator topology (WOT). Therefore $p_n(T)K(\lambda, \cdot)\zeta \rightarrow AK(\lambda, \cdot)\zeta$ weakly. But $p_n(T)K(\lambda, \cdot)\zeta = p_n(\lambda)K(\lambda, \cdot)\zeta \rightarrow \psi(\lambda)K(\lambda, \cdot)\zeta$ weakly. Hence $AK(\lambda, \cdot)\zeta = \psi(\lambda)K(\lambda, \cdot)\zeta$. From this we conclude that $A = X$, so $X \in W(T)$. Therefore $\text{Alg Lat}(T) \subset W(T)$ and T is reflexive. $\square$

Note that in Theorem 2 we can replace the assumption that $\sigma(T) = \bar{\Omega}$ is a spectral set by the requirement that $\|p(T)\| \leq \|p\|_\Omega$ for every polynomial p and the result is still true. In case T is a contraction and $\sigma(T) = \bar{\mathbf{D}}$ von Neumann's inequality holds. That is, $\|p(T)\| \leq \|p\|_{\mathbf{D}}$ for every polynomial p . We therefore have the following

Corollary 1. *If T is a contraction in $\mathcal{B}_n(\mathbf{D})$ then T is reflexive.*

4. CERTAIN BILATERAL WEIGHTED SHIFTS ARE REFLEXIVE

A unilateral (bilateral) weighted shift on a Hilbert space $\mathcal{H}$ is an operator T from $\mathcal{H}$ to itself given by $Te_n = w_n e_{n+1}$, where $\{e_n\} = \{e_n\}_{n=0}^\infty$ ($\{e_n\}_{n=-\infty}^\infty$) is an orthonormal basis for $\mathcal{H}$ and $\sup_n |w_n| < \infty$.

In his survey article, Shields [11] investigated the properties of weighted shifts. In particular, Question 18 of this article enquires which weighted shifts are reflexive. We shall discuss a class of bilateral shifts that are reflexive.

There is another way of viewing the weighted shifts, which is discussed in [11].

Let $\beta(n)$ be a sequence of positive numbers with $\beta(0) = 1$. We consider the space of sequences $f = \{\hat{f}(n)\}$ such that

$$\|f\|^2 = \|f\|_\beta^2 = \sum |\hat{f}(n)|^2 [\beta(n)]^2 < \infty.$$

The notation $f(z) = \sum \hat{f}(n)z^n$ shall be used whether or not the series converges for any value of z . If n ranges over nonnegative integers, these are formal power series; otherwise, they are formal Laurent series. Let $H^2(\beta)$ ($L^2(\beta)$) denote the space of such formal power series (Laurent series).

Let T be an injective unilateral (bilateral) weighted shift operator. It is well known that T can be represented as the operator of multiplication by z , M_z on $H^2(\beta)$ ($L^2(\beta)$) for some β . Let $H^\infty(\beta)$ ($L^\infty(\beta)$) denote the set of formal power series (Laurent series) $\phi(z) = \sum \hat{\phi}(n)z^n$ such that $\phi H^2(\beta) \subset H^2(\beta)$ ($\phi L^2(\beta) \subset L^2(\beta)$).

The relation between $\{w_n\}$ and β is $w_n = \beta(n+1)/\beta(n)$ (integer n), $\beta(n) = w_0 \cdots w_{n-1}$ ($n > 0$), $\beta(0) = 1$, and $\beta(-n) = (w_{-1} \cdots w_{-n})^{-1}$ ($n > 0$).

If T is an injective bilateral weighted shift with weight sequence $\{w_n\}$ then define

$$\begin{aligned} r_1^+(T) &= \lim_{n \rightarrow \infty} \left[\inf_{j \geq 0} \beta(n, j) \right]^{1/n}, & r^+(T) &= \lim_{n \rightarrow \infty} \left[\sup_{j \geq 0} \beta(n, j) \right]^{1/n}, \\ r_1^-(T) &= \lim_{n \rightarrow \infty} \left[\inf_{j < 0} \beta(-n, j) \right]^{1/n}, & r^-(T) &= \lim_{n \rightarrow \infty} \left[\sup_{j < 0} \beta(-n, j)^{-1} \right]^{1/n}, \end{aligned}$$

where $\beta(n, j) = \beta(n+j)/\beta(j)$, and

$$\begin{aligned} r_2^+(T) &= \lim_{n \rightarrow \infty} \inf (w_0 \cdots w_{n-1})^{1/n}, \\ r_3^+(T) &= \lim_{n \rightarrow \infty} \sup (w_0 \cdots w_{n-1})^{1/n}, \\ r_2^-(T) &= \lim_{n \rightarrow \infty} \inf (w_{-1} \cdots w_{-n})^{1/n}, \\ r_3^-(T) &= \lim_{n \rightarrow \infty} \sup (w_{-1} \cdots w_{-n})^{1/n}. \end{aligned}$$

Then $r_1^- \leq r_2^- \leq r_3^- \leq r^-$ and $r_1^+ \leq r_2^+ \leq r_3^+ \leq r^+$. We also let $r(T)$ denote the spectral radius of T and $r_1(T) = \lim_{n \rightarrow \infty} (m(T^n))^{1/n}$ where $m(A) = \inf\{\|Ax\| : \|x\| = 1\}$ for every operator A .

Now let T be an injective bilateral weighted shift represented as M_z on $L^2(\beta)$. Then the commutant $\{T\}'$ of T is $L^\infty(\beta)$ (where each $\phi \in L^\infty(\beta)$ is identified with the operator of multiplication by ϕ , M_ϕ on $L^2(\beta)$) [11]. Assume further that T is invertible. If $\|T^{-1}\|^{-1} = \|T\|$ then $L^\infty(\beta) = L^\infty(m)$

where m is the Lebesgue measure on $|z| = \|T\|$ [11, p. 80, Theorem 10'(vii)]. In this case $L^2(\beta) = L^2(m)$ and T is a scalar multiple of the (unweighted) bilateral shift. Hence T is reflexive.

Assume $\|T^{-1}\|^{-1} < \|T\|$, $r_1^- = r^- = \|T^{-1}\|^{-1}$, $r_1^+ = r^+ = \|T\|$, and set $\Omega = \{z : \|T^{-1}\|^{-1} < |z| < \|T\|\}$. Then $T^* \in \mathcal{B}_1(\Omega)$ by [10, Theorem 2.2]. Furthermore $L^\infty(\beta)$ coincides, as an algebra with the space of bounded analytic functions in Ω denoted by $H^\infty(\Omega)$. Hence $\{T\}' = L^\infty(\beta) = H^\infty(\Omega)$.

We now show the following

Proposition 2. *Let T be an invertible bilateral weighted shift such that $\|T^{-1}\|^{-1} < \|T\|$, $r_1^- = r^- = \|T^{-1}\|^{-1}$ and $r_1^+ = r^+ = \|T\|$. Then T is reflexive.*

Proof. Let $X \in \text{Alg Lat}(T)$. Then $X \in \{T\}'$ by Lemma 1. Hence $X = M_\phi$ for some $\phi \in H^\infty(\Omega)$. Since $\phi \in H^\infty(\Omega)$, we can write

$$\phi(z) = \sum_{-\infty}^{\infty} a_n z^n = \sum_0^{\infty} a_n z^n + \sum_{-\infty}^{-1} a_n z^n = \phi_1(z) + \phi_2(z)$$

where ϕ_1 is bounded and analytic in $|z| < \|T\|$, and ϕ_2 is bounded and analytic in $|z| > \|T^{-1}\|^{-1}$ [11, Lemma, p. 81]. Since ϕ_1 can be approximated by a sequence of polynomials, we conclude that $M_{\phi_1} \in W(T)$. To see this let $\{p_n\}$ be a sequence of polynomials uniformly bounded in $|z| < \|T\|$ converging pointwise to ϕ_1 . By von Neumann's inequality

$$\|p_n(T)\| \leq \sup\{|p_n(z)| : |z| < \|T\|\}.$$

Passing to a subsequence, if necessary, we may assume that $p_n(T) \rightarrow A$ (WOT) for some operator A . Now since every point in Ω is a bounded point evaluation we conclude that

$$M_{p_n}^* k_w = \overline{p_n(w)} k_w \rightarrow \overline{\phi_1(w)} k_w = M_{\phi_1}^* k_w$$

for every $w \in \Omega$. On the other hand, $M_{p_n}^* k_w \rightarrow A^* k_w$ weakly. Therefore, $A^* k_w = M_{\phi_1}^* k_w$. Since the linear span of k_w , $w \in \Omega$, is dense in $L^2(\beta)$, we conclude that $M_{\phi_1} = A \in W(T)$. We now show that $\phi_2 = 0$. Then $X = M_\phi = M_{\phi_1} \in W(T)$ and the proof is complete.

To see this note that $H^2(\beta)$ is invariant under T so it is invariant under X and M_{ϕ_1} . Hence $\phi_2 = \phi - \phi_1 = X1 - M_{\phi_1} 1 \in H^2(\beta)$. If $\phi_2 \neq 0$ then $\hat{\phi}_2(k) \neq 0$ for some $k > 0$. Since ϕ_2 is bounded and analytic in $|z| > \|T^{-1}\|^{-1}$, there is a sequence of polynomials $\{p_n(1/z)\}$ in $1/z$ uniformly bounded in $|z| > \|T^{-1}\|^{-1}$ and converging pointwise to ϕ_2 . Therefore $p_n(T^{-1}) \rightarrow M_{\phi_2}$ (WOT). To see this note that

$$M_{p_n(1/z)}^* k_w = \overline{p_n(1/w)} k_w \rightarrow \overline{\phi_2(w)} k_w = M_{\phi_2}^* k_w.$$

Hence $M_{p_n(1/z)}^* f \rightarrow M_{\phi_2}^* f$ for every f in the linear span of k_w , $w \in \Omega$. Now by [11, Theorem 10' vii(b), p. 80] we conclude that

$$\|M_{p_n(1/z)}\| \leq c \sup\{|p_n(z)|: \|T^{-1}\|^{-1} < |z| < \|T\|\}$$

where c is a constant. Therefore $\{\|M_{p_n(1/z)}\|\}$ is uniformly bounded and hence $M_{p_n(1/z)}^* f \rightarrow M_{\phi_2}^* f$ for every f in $L^2(\beta)$. We have actually shown that $p_n(T^{-1}) \rightarrow M_{\phi_2}$ (SOT). Therefore

$$\langle p_n(T^{-1})1, f_k \rangle \rightarrow \langle M_{\phi_2}1, f_k \rangle$$

where $f_k(z) = z^k$. That is, $\hat{p}_n(1/z)(k) \rightarrow \hat{\phi}_2(k)$ as $n \rightarrow \infty$. But the left-hand side is zero and the right-hand side is nonzero. This is a contradiction. Hence $\phi_2 = 0$. $\square$

The following result of Shields [11, Proposition 15, p. 72] will provide us with a large class of operators satisfying the hypothesis of Proposition 4.1.

Proposition 3. *Let T be a bilateral weighted shift with weight sequence $\{w_n\}$. If $w_n \rightarrow d^+$ ($n \rightarrow \infty$) and $w_n \rightarrow d^-$ ($n \rightarrow -\infty$), then*

$$r_1^- = \cdots = r^- = \|T^{-1}\|^{-1} = d^-, \quad r_1^+ = \cdots = r^+ = \|T\| = d^+.$$

5. FINITELY CONNECTED DOMAINS

Suppose $\mathcal{H}$ is a Hilbert space of functions analytic on a plane domain Ω , and suppose that for each point $\lambda \in \Omega$ the linear functional of evaluation at λ is bounded on $\mathcal{H}$. We further assume that $\mathcal{H}$ contains the constant functions and multiplication by the independent variable z defines a bounded linear operator M_z on $\mathcal{H}$. Note that by the continuity of point evaluations and by the Riesz representation theorem, there exists for each $\lambda \in \Omega$ a unique function $k_\lambda \in \mathcal{H}$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle \quad (f \in \mathcal{H}).$$

The function k_λ is the *reproducing kernel* for the point λ , and its norm is the same as that of the corresponding evaluation functional.

Now let L be a compact subset of Ω and choose $f \in \mathcal{H}$. Then $|f(\lambda)| \leq C_f$, $\lambda \in L$, where C_f is a constant depending on f . This means $|\langle f, k_\lambda \rangle| \leq C_f$, $\lambda \in L$. By the principle of uniform boundedness we have $\|k_\lambda\| \leq C$, $\lambda \in L$, for some constant C . We conclude that convergence in $\mathcal{H}$ implies uniform convergence on compact subsets of Ω .

A complex-valued function ϕ on Ω for which $\phi f \in \mathcal{H}$ for every $f \in \mathcal{H}$ is called a *multiplier* of $\mathcal{H}$, and the collection of all these multipliers is denoted $\mathcal{M}(\mathcal{H})$. Each multiplier ϕ of $\mathcal{H}$ determines a multiplication operator M_ϕ on $\mathcal{H}$ by the formula $M_\phi f = \phi f$ ($f \in \mathcal{H}$). An application of the closed graph theorem shows that M_ϕ is a bounded operator on $\mathcal{H}$. It is well known that each multiplier is a bounded analytic function on Ω [12, Lemma 3, p. 782].

Actually $\|\phi\|_{\Omega} \leq \|M_{\phi}\|$. It is also easy to see that if ϕ is a multiplier and $\lambda \in \Omega$, then $M_{\phi}^* k_{\lambda} = \overline{\phi(\lambda)} k_{\lambda}$.

We now assume that Ω is a finitely connected domain. It is well known that Ω is conformally equivalent to a circular domain [2]. By a circular domain we mean any domain that is obtained by removing a finite number of disjoint closed subdisks from the open unit disk. So we let $\Omega = \mathbf{D} \setminus (D_1^- \cup D_2^- \cup \dots \cup D_N^-)$ where $D_i^- = \{z : |z - z_i| \leq r_i\}$ ($i = 1, \dots, N$) are disjoint closed subdisks of the open unit disk $\mathbf{D}$. We can choose circles $\gamma_i = \{z : |z - z_i| = r_i + \varepsilon_i\}$ ($i = 1, \dots, N$) and $\gamma_0 = \{z : |z| = 1 - \varepsilon_0\}$ lying in Ω and concentric to the boundary circles of Ω so that they do not meet each other. We denote $\Omega_i = \mathbf{C} \setminus D_i^-$ ($i = 1, \dots, N$).

It is proved in [2] that

$$H^{\infty}(\Omega) = H^{\infty}(\mathbf{D}) + H_0^{\infty}(\Omega_1) + \dots + H_0^{\infty}(\Omega_N)$$

where the subscript zero means that the corresponding functions vanish at ∞ .

We now prove the following

Theorem 3. *Let Ω be a finitely connected plane domain each point of which is a bounded point evaluation for a Hilbert space $\mathcal{H}$ of functions analytic on Ω that contains the constant functions and admits multiplication by the independent variable z , M_z , as a bounded operator. Furthermore, suppose that the map $\phi \rightarrow M_{\phi}$ from $\mathcal{M}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$ is an isometry. Then M_z is reflexive.*

Proof. Set $\mathcal{H}_0 = \{f \in \mathcal{H} : \int_{\gamma_0} \zeta^n f(\zeta) d\zeta = 0, n = 0, 1, 2, \dots\}$. Note that $\mathcal{H}_0$ is a subspace of $\mathcal{H}$. To see that $\mathcal{H}_0$ is closed let $\{f_k\}$ be a sequence in $\mathcal{H}_0$ such that $f_k \rightarrow f$ in $\mathcal{H}$. Then $f_k \rightarrow f$ uniformly on compact subsets of Ω . It is now easy to see that $f \in \mathcal{H}_0$. Clearly $\mathcal{H}_0$ is invariant under M_z and contains the constants.

We now show that every function in $\mathcal{H}_0$ is analytic in $\mathbf{D}$. To see this let $f \in \mathcal{H}_0$. Then applying the Cauchy integral formula we can write $f = f_0 + f_1 + \dots + f_N$ where $f_0 \in H(\mathbf{D})$ and $f_i \in H_0(\Omega_i)$ ($i = 1, \dots, N$). Set $g = f_1 + \dots + f_N$. Then g is analytic in $\text{ext}(\gamma'_0)$ the unbounded component of $\mathbf{C} \setminus \gamma'_0$ where the circle γ'_0 is chosen sufficiently close to γ_0 with smaller radius so that γ_0 lies in $\text{ext}(\gamma'_0)$. We can write

$$g(z) = \sum_{n=-\infty}^{-1} a_n z^n, \quad z \in \text{ext}(\gamma'_0).$$

Since $f_0 \in H(\mathbf{D})$, we have $\int_{\gamma_0} \zeta^n f_0(\zeta) d\zeta = 0$, $n = 0, 1, 2, \dots$; therefore, $\int_{\gamma_0} \zeta^n g(\zeta) d\zeta = 0$, $n = 0, 1, 2, \dots$. From this it follows that $g(z) = 0$, $z \in \text{ext}(\gamma'_0)$. Hence $g \equiv 0$. Therefore $f = f_0$ is analytic in $\mathbf{D}$.

Now let $X \in \text{Alg Lat}(M_z)$. Then by an argument similar to the proof of Lemma 1 we can show that $X = M_{\phi}$ for some multiplier ϕ . A direct proof is also possible by noting that the one-dimensional span of $\{k_{\lambda}\}$ is invariant under M_z^* and hence $X^* k_{\lambda} = \overline{\phi(\lambda)} k_{\lambda}$. This too shows that $X = M_{\phi}$. Since $\mathcal{H}_0$ is invariant under M_z , we have $X\mathcal{H}_0 \subset \mathcal{H}_0$, so $X1 = \phi \in \mathcal{H}_0$ and hence

$\phi \in H(\mathbf{D})$. Because ϕ is a multiplier, $\phi \in H^\infty(\Omega)$. Applying the maximum modulus principle, we conclude that $\phi \in H^\infty(\mathbf{D})$ and $\|\phi\|_{\mathbf{D}} = \|\phi\|_{\Omega}$.

Let $\{p_n\}$ be a sequence of polynomials such that $\sup\|p_n\|_{\mathbf{D}} < \infty$ and $p_n(z) \rightarrow \phi(z)$, $z \in \mathbf{D}$. Then $\|M_{p_n}\| = \|p_n\|_{\Omega} = \|p_n\|_{\mathbf{D}} < M$ where M is a constant. By passing to a subsequence we may assume that $M_{p_n} \rightarrow A$ (WOT). Since $M_{p_n}^* k_\lambda = \overline{p_n(\lambda)} k_\lambda \rightarrow \overline{\phi(\lambda)} k_\lambda$, $\lambda \in \Omega$, we conclude that $A = X$. Hence $X \in W(T)$. This shows that T is reflexive. $\square$

Remark. In Theorem 3 we assumed that the map $\phi \rightarrow M_\phi$ from $\mathcal{M}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$ is an isometry. We may replace this by the assumption that $\|M_\phi\| \leq c\|\phi\|_{\Omega}$ where c is a constant, or we may suppose that $\sigma(M_z) = \overline{\Omega}$ is a spectral set for M_z . Even the assumption $\|M_p\| = \|p\|_{\Omega}$ or $\|M_p\| \leq c\|p\|_{\Omega}$ for every polynomial p would be sufficient for our purposes.

Corollary 2. *Let $\mathcal{H}$ be a Hilbert space of functions analytic on a finitely connected domain Ω such that the point evaluations are bounded, $\mathcal{H}$ contains the constants, and $\|M_z\| \leq 1$. Then M_z is reflexive.*

Proof. By von Neumann's inequality $\|M_p\| \leq \|p\|_{\mathbf{D}}$. But since $\|p\|_{\mathbf{D}} = \|p\|_{\Omega}$ for every polynomial p , the conclusion now follows from the Remark. $\square$

REFERENCES

1. H. Bercovici, C. Foias, J. Langsam, and C. Pearcy, *(BCP)-operators are reflexive*, Michigan Math. J. **29** (1982), 371–379.
2. B. Chevreau, C. Pearcy, and A. Shields, *Finitely connected domains G , representations of $\mathcal{H}^\infty(G)$, and invariant subspaces*, J. Operator Theory **6** (1981), 375–405.
3. M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math. **141** (1978), 187–261.
4. R. Cutro and N. Salinas, *Generalized Bergman kernels and the Cowen-Douglas theory*, Amer. J. Math. **106** (1984), 447–488.
5. J. A. Deddens, *Every isometry is reflexive*, Proc. Amer. Math. Soc. **28** (1971), 509–512.
6. J. A. Deddens and P. A. Fillmore, *Reflexive linear transformations*, Linear Algebra Appl. **10** (1975), 89–93.
7. T. Gamelin, *Uniform algebras*, Chelsea, New York, 1984.
8. R. Olin and J. Thomson, *Algebras of subnormal operators*, J. Funct. Anal. **37** (1980), 271–301.
9. D. Sarason, *Invariant subspaces and unstarred operator algebras*, Pacific J. Math. **17** (1966), 511–517.
10. K. Seddighi, *Essential spectra of operators in the class $\mathcal{B}_n(\Omega)$* , Proc. Amer. Math. Soc. **87** (1983), 453–458.
11. A. Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory, Math. Surveys Monographs, vol. 13, Amer. Math. Soc., Providence, RI, 1974, pp. 49–128.
12. A. Shields and L. Wallen, *The commutants of certain Hilbert space operators*, Indiana Univ. Math. J. **20** (1971), 777–788.

DEPARTMENT OF MATHEMATICS, SHIRAZ UNIVERSITY, SHIRAZ, IRAN