MAXIMAL DOUGLAS SUBALGEBRAS AND MINIMAL SUPPORT POINTS

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ABSTRACT. Let B denote a Douglas algebra. Then B has a maximal Douglas subalgebra if and only if the set of points outside the maximal ideal space of B has a minimal support point.

1. INTRODUCTION

Let H^∞ be the space of bounded analytic functions on the open unit disc D , and let L^∞ be the space of bounded measurable functions on the unit circle ∂D with respect to the Lebesgue measure $d\theta/2\pi$. Identifying H^∞ with these boundary functions, we may consider that H^∞ is an essentially sup-norm closed subalgebra of L^∞ . An essentially sup-norm closed subalgebra B of L^∞ containing H^∞ is called a *Douglas algebra*. We denote by C the space of continuous functions on ∂D . Then it is well known that $H^\infty + C$ is a Douglas algebra and contains H^∞ as a maximal Douglas subalgebra [10]. Also in [4], Guillory, Izuchi, and Sarason showed that there is a Douglas algebra B such that $H^\infty + C \subsetneq B \subsetneq L^\infty$ and B has a maximal Douglas subalgebra. On the other hand, Sundberg [11] proved that L^∞ does not have a maximal Douglas subalgebra. In this paper, we answer to a question when a given Douglas algebra has a maximal Douglas subalgebra.

For a Douglas algebra B , we let $M(B)$ denote the maximal ideal space of B . A Gelfand transform of f in B is denoted by the same symbol f . $M(B)$ may be identified with a closed subset of $M(H^\infty)$. Then $M(L^\infty)$ becomes the Shilov boundary for B and $M(H^\infty + C) = M(H^\infty) \setminus D$. For a point x in $M(H^\infty)$, we denote by μ_x its representing measure on $M(L^\infty)$, that is, $f(x) = \int_{M(L^\infty)} f d\mu_x$ for every f in H^∞ . We put

$$H_{\text{supp } \mu_x}^\infty = \{f \in L^\infty; f|_{\text{supp } \mu_x} \in H_{\text{supp } \mu_x}^\infty\};$$

then $H_{\text{supp } \mu_x}^\infty$ is a Douglas algebra and

$$M(H_{\text{supp } \mu_x}^\infty) = M(L^\infty) \cup \{\zeta \in M(H^\infty); \text{supp } \mu_\zeta \subset \text{supp } \mu_x\}.$$

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For a subset Ω of $M(H^\infty)$, a point x in Ω is called a *minimal support point* for Ω if there are no points y in Ω such that $\text{supp } \mu_y \subsetneq \text{supp } \mu_x$. The following is our main theorem.

Theorem 1. *Let B be a Douglas algebra. Then B has a maximal Douglas subalgebra if and only if $M(H^\infty) \setminus M(B)$ has a minimal support point. If x is a minimal support point for $M(H^\infty) \setminus M(B)$, then $B \cap H_{\text{supp } \mu_x}^\infty$ is a maximal Douglas subalgebra of B , and every maximal Douglas subalgebra is given by this form.*

In §2, we prove Theorem 1. In §3, we give some applications of Theorem 1.

2. PROOF OF THEOREM 1

A function ψ in H^∞ with $|\psi| = 1$ on $M(L^\infty)$ is called *inner*. The famous Chang-Marshall theorem [1, 9] says that if B is a Douglas algebra then B is generated by H^∞ and complex conjugates of inner functions ψ with $\bar{\psi} \in B$ and $M(B)$ is the set of points x in $M(H^\infty)$ such that $|\psi(x)| = 1$ for every inner function ψ with $\bar{\psi} \in B$, that is, $M(B) = \{x \in M(H^\infty); B|_{\text{supp } \mu_x} = H_{\text{supp } \mu_x}^\infty\}$. We use these facts frequently. To prove our theorem, we need the following unpublished result of Sarason (see [3, Proposition 3.4]).

Lemma 1. *Let B_1 and B_2 be Douglas algebras. Then $M(B_1 \cap B_2) = M(B_1) \cup M(B_2)$.*

Proof of Theorem 1. Let A be a maximal Douglas subalgebra of B . Since $M(B) \subsetneq M(A)$, take a point x in $M(A) \setminus M(B)$. To prove that x is a minimal support point for $M(H^\infty) \setminus M(B)$, suppose not. Then there exists y in $M(H^\infty) \setminus M(B)$ such that $\text{supp } \mu_y \subsetneq \text{supp } \mu_x$. Let $C = B \cap H_{\text{supp } \mu_y}^\infty$. Since $x \in M(A)$, it follows that $A|_{\text{supp } \mu_x} = H_{\text{supp } \mu_x}^\infty$, so that $A|_{\text{supp } \mu_y} = H_{\text{supp } \mu_y}^\infty$. Therefore we have $A \subset C \subset B$. By Lemma 1,

$$M(C) = M(B) \cup \{\zeta \in M(H^\infty); \text{supp } \mu_\zeta \subset \text{supp } \mu_y\}.$$

Since $y \in M(C)$ and $y \notin M(B)$, it follows that $M(B) \subsetneq M(C)$ and $C \subsetneq B$. Since $x \notin M(C)$ and $x \in M(A)$, it follows that $M(C) \subsetneq M(A)$ and $A \subsetneq C$. Hence $A \subsetneq C \subsetneq B$, contradicting the maximality of A .

Since $A \subset H_{\text{supp } \mu_x}^\infty$, we have $A \subset B \cap H_{\text{supp } \mu_x}^\infty$. Since $x \notin M(B)$ and $x \in M(B \cap H_{\text{supp } \mu_x}^\infty)$, it follows that $B \cap H_{\text{supp } \mu_x}^\infty \subsetneq B$. By the maximality of A , we get $A = B \cap H_{\text{supp } \mu_x}^\infty$.

To show the converse, let x be a minimal support point for $M(H^\infty) \setminus M(B)$ and let $A = B \cap H_{\text{supp } \mu_x}^\infty$. Then $A \subset B$ and $M(A) = M(B) \cup \{\zeta \in M(H^\infty); \text{supp } \mu_\zeta \subset \text{supp } \mu_x\}$. Since $x \in M(A)$ and $x \notin M(B)$, we have $A \subsetneq B$. Since x is a minimal support point for $M(H^\infty) \setminus M(B)$, if $\text{supp } \mu_\zeta \subsetneq \text{supp } \mu_x$ then $\zeta \in M(B)$. Hence

$$M(A) = M(B) \cup \{\zeta \in M(H^\infty); \text{supp } \mu_\zeta = \text{supp } \mu_x\}.$$

Let Y be a Douglas algebra such that $A \subsetneq Y \subset B$. By Chang-Marshall's theorem, there is an inner function ψ such that $\bar{\psi} \notin A$ and $\bar{\psi} \in Y \subset B$. Then $|\psi| = 1$ on $M(B)$ and $|\psi(x)| < 1$. Hence $|\psi| < 1$ on $M(A) \setminus M(B)$. Since $M(B) \subset M(Y) \subset M(A)$, it follows that $M(B) = M(Y)$. Consequently $B = Y$ and this implies that A is a maximal Douglas subalgebra of B .

Remark. By the above proof, if $A \subsetneq B$ then A is a maximal Douglas subalgebra of B if and only if $\text{supp } \mu_x = \text{supp } \mu_y$ for every x, y in $M(A) \setminus M(B)$.

3. APPLICATIONS

A Blaschke product b with zeros $\{z_n\}_n$ in D is called sparse if

$$\lim_{k \rightarrow \infty} \prod_{n: n \neq k} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| = 1.$$

The above condition implies that $\{z_n\}_n$ is interpolating for H^∞ , that is, for every bounded sequence $\{a_n\}_n$ there exists f in H^∞ such that $f(z_n) = a_n$ for every n . For every sequence $\{y_n\}_n$ in D with $|y_n| \rightarrow 1$, we can find a subsequence that satisfies the sparseness condition. For a function f in H^∞ , we put $Z(f) = \{x \in M(H^\infty + C); f(x) = 0\}$. For a subset F of L^∞ , we denote by $[F]$ the closed subalgebra of L^∞ generated by F .

Theorem 2. Let ψ be an inner function such that ψ is not continuous on ∂D . Then $[H^\infty, \bar{\psi}]$ has uncountably many maximal Douglas subalgebras.

Proof. We have $M([H^\infty, \bar{\psi}]) = \{\zeta \in M(H^\infty + C); |\psi(\zeta)| = 1\}$. For $\zeta \in M(H^\infty) \setminus M([H^\infty, \bar{\psi}])$, there is a point x in $Z(\psi)$ such that $\text{supp } \mu_x \subset \text{supp } \mu_\zeta$. Hence to study minimal support points for $M(H^\infty) \setminus M([H^\infty, \bar{\psi}])$ is the same as to study minimal support points for $Z(\psi)$, so we concentrate on $Z(\psi)$.

For x, y in $Z(\psi)$, we define the order as follows; $y \leq x$ if $\text{supp } \mu_y \subset \text{supp } \mu_x$, and $x = y$ if $\text{supp } \mu_y = \text{supp } \mu_x$. By Hoffman's unpublished result [7], if $\text{supp } \mu_y \cap \text{supp } \mu_x \neq \emptyset$ then $\text{supp } \mu_y \subset \text{supp } \mu_x$ or $\text{supp } \mu_y \supset \text{supp } \mu_x$. Hence $Z(\psi)$ becomes an ordered set. Let $\{x_\alpha\}_{\alpha \in \Lambda}$ be a totally ordered subset of $Z(\psi)$. We denote by F_α the closure of $\{x_\beta; \beta \in \Lambda, \beta \leq \alpha\}$ in $M(H^\infty)$. Then $\{F_\alpha\}_\alpha$ is a family of compact decreasing subsets of $Z(\psi)$. Therefore, there is a point x_0 in $\bigcap_\alpha F_\alpha$, and we have $\text{supp } \mu_{x_0} \subset \text{supp } \mu_{x_\alpha}$ and $x_0 \leq x_\alpha$. By Zorn's lemma, $Z(\psi)$ has a minimal element; hence $M(H^\infty) \setminus M([H^\infty, \bar{\psi}])$ has a minimal support point. By Theorem 1, $[H^\infty, \bar{\psi}]$ has a maximal Douglas subalgebra.

Let b be a sparse Blaschke product with zeros $\{z_n\}_n$ such that $|\psi(z_n)| \rightarrow 0$ ($n \rightarrow \infty$). By [5, p. 205], $Z(b)$ is contained in the closure of $\{z_n\}_n$ in $M(H^\infty)$. Then $Z(b) \subset Z(\psi)$. By the second paragraph, for each $\zeta \in Z(b)$ there is a minimal element ζ_0 in $Z(\psi)$ such that $\zeta_0 \leq \zeta$. By Theorem 1, $[H^\infty, \bar{\psi}]$ has maximal Douglas subalgebras $\{[H^\infty, \bar{\psi}] \cap H^\infty_{\text{supp } \mu_{\zeta_0}}; \zeta \in Z(b)\}$. By [4, p. 5], $\text{supp } \mu_\zeta \cap \text{supp } \mu_\lambda = \emptyset$ for distinct $\zeta, \lambda \in Z(b)$. Since $\text{supp } \mu_{\zeta_0} \subset \text{supp } \mu_\zeta$, it follows that $[H^\infty, \bar{\psi}] \cap H^\infty_{\text{supp } \mu_{\zeta_0}} \neq [H^\infty, \bar{\psi}] \cap H^\infty_{\text{supp } \mu_{\lambda_0}}$ for distinct $\zeta, \lambda \in Z(b)$. Since $Z(b)$ is homeomorphic to $\beta N \setminus N$ [5, p. 205] where βN is the Stone-Čech compactification of integers N , we get our assertion.

Theorem 3. Let B be a Douglas algebra with $B \subsetneq L^\infty$. Let ψ be an inner function with $\bar{\psi} \notin B$. Then $[B, \bar{\psi}]$ has a maximal Douglas subalgebra.

Proof. Let $x \in M(B) \setminus M([B, \bar{\psi}])$. Then $|\psi(x)| < 1$. By the proof of Theorem 2, there is a minimal support point y for $M(H^\infty) \setminus M([H^\infty, \bar{\psi}])$ such that $\text{supp } \mu_y \subset \text{supp } \mu_x$. To prove that y is a minimal support point for $M(H^\infty) \setminus M([B, \bar{\psi}])$, suppose that there is a point ζ in $M(H^\infty) \setminus M([B, \bar{\psi}])$ such that $\text{supp } \mu_\zeta \subsetneq \text{supp } \mu_y$. Since $x \in M(B)$, we have $y \in M(B)$, so that

$\zeta \in M(B)$. Since $\zeta \notin M([B, \bar{\psi}])$, $|\psi(\zeta)| < 1$. This contradicts the minimality of y for $M(H^\infty) \setminus M([H^\infty, \bar{\psi}])$. Hence y is a minimal support point. By Theorem 1, we get Theorem 3.

The following theorem shows that there exists a Douglas algebra that does not have any maximal Douglas subalgebras.

Theorem 4. *Let $\{\psi_n\}_{n=1}^\infty$ be a sequence of inner functions such that $\psi_{n+1} = 0$ on $\{\zeta \in M(H^\infty + C); |\psi_n(\zeta)| < 1\}$ for every n . Then $[H^\infty, \bar{\psi}_n; n = 1, 2, \dots]$ does not have a maximal Douglas subalgebra.*

Proof. Put $B = [H^\infty, \bar{\psi}_n; n = 1, 2, \dots]$. Let $x \in M(H^\infty) \setminus M(B)$. We shall prove that there is a point y in $M(H^\infty) \setminus M(B)$ such that $\text{supp } \mu_y \subsetneq \text{supp } \mu_x$. Since $x \notin M(B)$, it follows that $|\psi_n(x)| < 1$ for some n ; hence $\{\zeta \in M(H^\infty_{\text{supp } \mu_x}); |\psi_n(\zeta)| < 1\}$ is a nonvoid open subset of $M(H^\infty_{\text{supp } \mu_x})$. But it is not closed in $M(H^\infty_{\text{supp } \mu_x})$. For if it is closed in $M(H^\infty_{\text{supp } \mu_x})$, by the Shilov idempotent theorem, there is a function g in $H^\infty_{\text{supp } \mu_x}$ such that $g^2 = g$ and $\{\zeta \in M(H^\infty_{\text{supp } \mu_x}); g(\zeta) = 1\} = \{\zeta \in M(H^\infty_{\text{supp } \mu_x}); |\psi_n(\zeta)| = 1\}$. Since $g = 1$ on $M(L^\infty)$ and $|\psi_n(x)| \neq 1$, we have $0 = g(x) = \int_{M(L^\infty)} g d\mu_x = 1$; this is impossible.

Take a point y in the closure of $\{\zeta \in M(H^\infty_{\text{supp } \mu_x}); |\psi_n(\zeta)| < 1\}$ such that $|\psi_n(y)| = 1$. Since $\text{supp } \mu_\zeta \subset \text{supp } \mu_x$ for ζ in $M(H^\infty_{\text{supp } \mu_x})$ with $|\psi_n(\zeta)| < 1$, we have $\text{supp } \mu_y \subset \text{supp } \mu_x$. Since ψ_n is constant on $\text{supp } \mu_y$ and ψ_n is not constant on $\text{supp } \mu_x$, we have $\text{supp } \mu_y \subsetneq \text{supp } \mu_x$. Since $\psi_{n+1} = 0$ on $\{\zeta \in M(H^\infty + C); |\psi_n(\zeta)| < 1\}$, we have $\psi_{n+1}(y) = 0$, so that $y \notin M(B)$. Hence every point in $M(H^\infty) \setminus M(B)$ is not a minimal support point for $M(H^\infty) \setminus M(B)$. By Theorem 1, we get Theorem 4.

Corollary 1. *There is a function f in L^∞ such that $[H^\infty, f]$ does not have a maximal Douglas subalgebra.*

Proof. By [11] there is a sequence of inner functions $\{\psi_n\}_n$ that satisfy the condition of Theorem 4. Take f in L^∞ such that $[H^\infty, f] = [H^\infty, \bar{\psi}_n; n = 1, 2, \dots]$ (see [8, Lemma 2.2]). Then f satisfies our assertion.

For a Douglas algebra B , a Douglas algebra A with $B \subsetneq A$ is called a *minimal superalgebra of B* if there are no Douglas algebras J such that $B \subsetneq J \subsetneq A$; that is, B is a maximal Douglas subalgebra of A . If A is a minimal superalgebra of B , then $A = [B, \bar{\psi}]$ for some inner function ψ . The following theorem is a restatement of Theorem 1 and Remark.

Theorem 5. *Let B be a Douglas algebra. Then B has a minimal superalgebra if and only if there is an inner function ψ , $\bar{\psi} \notin B$ such that $\text{supp } \mu_x = \text{supp } \mu_y$ for every x, y in $\{\zeta \in M(B); |\psi(\zeta)| < 1\}$. Under the above situation, $[B, \bar{\psi}]$ becomes a minimal superalgebra of B .*

Corollary 2. *For every function f in L^∞ with $f \notin H^\infty$, $[H^\infty, f]$ does not have any minimal superalgebras.*

Proof. Put $B = [H^\infty, f]$. By [8, Lemma 2.2], there is a sequence of inner functions $\{\psi_n\}_n$ such that $B = [H^\infty, \bar{\psi}_n; n = 1, 2, \dots]$. Then $M(B)$ is a closed G_δ -subset of $M(H^\infty)$. Let g be a nonnegative continuous function on $M(H^\infty)$ such that $g = 1$ on $M(B)$ and $0 \leq g < 1$ on $M(H^\infty) \setminus M(B)$. To prove our assertion, let ψ be an inner function such that $\bar{\psi} \notin B$. Then

$Z(\psi) \cap M(B) \neq \emptyset$. By the corona theorem (see [2, p. 318]), there is a sequence $\{z_n\}_n$ in D such that $g(z_n) \rightarrow 1$ and $\psi(z_n) \rightarrow 0$. Let b be a Blaschke product with zeros $\{z_n\}_n$. We may assume that b is sparse. Then we have $Z(b) \subset M(B) \cap Z(\psi)$ and $\text{supp } \mu_x \neq \text{supp } \mu_y$ for x, y in $Z(b)$, $x \neq y$. By Theorem 5, we get our assertion.

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