INTERPOLATION BETWEEN WEIGHTED HARDY SPACES

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ABSTRACT. We prove that $H^p(w_0^{1-s}w_1^s)$ is an interpolation space of exponent s between $H^p(w_0)$ and $H^p(w_1)$ if and only if $\log(w_1/w_0)$ is in BMO. If $\log(w_1/w_0)$ fails to be in BMO, $H^p(w_0^{1-s}w_1^s)$ can still be an interpolation space, provided the range of w has sufficiently large gaps.

INTRODUCTION

The first theorem on the interpolation of linear operators is due to Riesz [Ri], who showed that if T is a linear operator that is continuous from $L^{p_0}(\mu)$ to $L^{p_0}(\mu)$ and from $L^{p_1}(\mu)$ to $L^{p_1}(\mu)$, then it is continuous from $L^p(\mu)$ to $L^p(\mu)$, for all $1 \leq p_0 \leq p \leq p_1 \leq \infty$. Many interpolation theorems have been proved since then; we mention only a result of Stein [St] that allows the measure to change: if T is continuous from $L^p(w_0\mu)$ to $L^p(w_0\mu)$, and from $L^p(w_1\mu)$ to $L^p(w_1\mu)$, then it is continuous from $L^p(w_0^{1-s}w_1^s\mu)$ to itself, for every $0 \leq s \leq 1$.

In this paper we consider the question of when linear operators defined and continuous on two weighted Hardy spaces must also be continuous on some other interpolating weighted Hardy space. By a *weight* we shall always mean a nonnegative measurable function w on the unit circle that satisfies $\int_{\mathbb{T}} |\log(w)| d\sigma < \infty$, where σ is a normalised Lebesgue measure. If w is also integrable, the weighted Hardy space $H^p(w)$ is the closure in $L^p(w\sigma)$ of the (analytic) polynomials, for $1 \leq p < \infty$. This is isometrically isomorphic to the space ψH^p , where H^p is the classical Hardy space (i.e., $w \equiv 1$), ψ is the outer function with modulus $w^{-1/p}$, and the norm of ψf in ψH^p is defined to be the norm of f in H^p (see [Ga] for the definition of an outer function). For an arbitrary weight, one can always find such an outer function (because of the log-integrability assumption), and so we define $H^p(w)$ to be ψH^p . For simplicity, we shall always assume that $p \geq 1$.

Given any two Banach spaces X_0 and X_1 that are 'compatible', in the sense of both being continuously embedded in some topological vector space, there are two principal ways of defining a family of intermediate spaces X_s such that any linear operator that is continuous from X_0 to X_0 and from X_1 to X_1 is

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also continuous from X_s to X_s , $0 < s < 1$: these are the so-called real and complex methods (see [BL] for details). Most interpolation theorems are proved by showing that the desired space is isomorphic to one of these X_s 's. Both these methods have the property that the norm of an operator T on X_s is less than or equal to $\|T\|_{X_0}^{1-s} \|T\|_{X_1}^s$. By analogy with Stein's theorem, one might expect that the space $H^p(w_0^{1-s} w_1^s)$ would be isomorphic to an interpolation space between $H^p(w_0)$ and $H^p(w_1)$ obtained by one of these methods. In §1 we prove that this is true if and only if $\log(w_1/w_0)$ is in the space BMO of functions of bounded mean oscillation (see [Ga] for a definition of BMO). If $\log(w_1/w_0)$ is not in BMO, we show that $H^p(w_0^{1-s} w_1^s)$ is not an interpolation space of exponent s (i.e., it does not satisfy $\|T\|_{H^p(w_0^{1-s} w_1^s)} \leq C \|T\|_{H^p(w_0)}^{1-s} \|T\|_{H^p(w_1)}^s$ for all linear operators bounded on both $H^p(w_0)$ and $H^p(w_1)$).

In §2 we give an example of a weight w whose logarithm is not in BMO, but such that $H^2(\sqrt{w})$ is an interpolation space between H^2 and $H^2(w)$. The example is a step function—such an example is necessary, as this behaviour cannot occur if the essential range of $\log(w)$ does not have large gaps (for example if the sizes of the components of the complement are bounded).

In §3 we consider the question of when, for any fixed Banach space Y , the continuity of a linear map T from $H^p(w_0)$ to Y and from $H^p(w_1)$ to Y implies the continuity of T from $H^p(w_0^{1-s} w_1^s)$ to Y . The answer is that this holds if and only if $H^p(w_0^{1-s} w_1^s)$ is contained in $H^p(w_0) + H^p(w_1)$. We show that this fails if w_1/w_0 has one-sided limits of zero and infinity at the same point.

In §4 we apply the positive result of §1 to a problem raised in [DM^{CC}], namely, determining which Toeplitz operators are bounded on $H^2(|e^{i\theta} - 1|)$. This was the problem that originally inspired this paper.

We remark that there is another, nonequivalent, way of defining weighted Hardy spaces; for details, and for interpolation theorems proved for these spaces, see [ST].

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Let us fix weights w_0 and w_1 , a real number $p \geq 1$, and a number s strictly between 0 and 1. Any interpolation theorem between $H^p(w_0)$ and $H^p(w_1)$ will depend only on the quotient $w = w_1/w_0$, so we shall assume that $w_0 = 1$, $w_1 = w$. For every positive real number t , define a pair of outer functions u_t and v_t by requiring that their moduli on the circle be given almost everywhere by

$$(1.1) \quad \begin{aligned} |u_t(e^{i\theta})| &= \min(1, tw^{\frac{1}{p}}(e^{i\theta})), \\ |v_t(e^{i\theta})| &= \min(1, 1/tw^{\frac{1}{p}}(e^{i\theta})). \end{aligned}$$

The solution of the interpolation problem depends on whether one can find a positive uniform lower bound on $|u_t(z)| + |v_t(z)|$ as z ranges over the unit disk $\mathbb{D}$; the following lemma asserts that the bound exists only if $\log(w)$ is in BMO.

Lemma 1.2. *Let u_t and v_t be defined as above. There exists $c > 0$ such that $|u_t(z)| + |v_t(z)| > c$ for all z in $\mathbb{D}$ and all $t > 0$ if and only if $\log(w)$ is in BMO.*

Proof. By [Ga, p. 225], $\log(w^{\frac{1}{p}})$ fails to be in BMO if and only if, for any $M > 0$, there exists z in $\mathbb{D}$ such that

$$(1.3) \quad \int_{\mathbb{T}} \left| \log(w^{\frac{1}{p}}(e^{i\theta})) - \int_{\mathbb{T}} \log(w^{\frac{1}{p}}(e^{i\phi})) P_z(\phi) d\sigma(\phi) \right| P_z(\theta) d\sigma(\theta) > M,$$

where $P_z(\cdot)$ is the Poisson kernel for z . The integral in (1.3) remains unchanged if w is replaced by tw ; choose t so that $\int \log(tw^{\frac{1}{p}}(e^{i\theta})) P_z(\theta) d\sigma(\theta)$ is zero. Then

$$\log |u_t(z)| = \int_{\{\theta : tw^{\frac{1}{p}}(e^{i\theta}) < 1\}} \log(tw^{\frac{1}{p}}(e^{i\theta})) P_z(\theta) d\sigma(\theta) < -\frac{M}{2}$$

and

$$\log |v_t(z)| = - \int_{\{\theta : tw^{\frac{1}{p}}(e^{i\theta}) > 1\}} \log(tw^{\frac{1}{p}}(e^{i\theta})) P_z(\theta) d\sigma(\theta) < -\frac{M}{2}.$$

So (1.3) implies the existence of $t > 0$ such that $|u_t(z)| + |v_t(z)| < 2e^{-M/2}$. The converse follows from a similar argument. $\square$

How small u and v can be made simultaneously controls the norm of the inclusion map from $H^p(w^s)$ into $H^p + H^p(w)$ (if the former space is not contained in the latter, we shall say the norm of the inclusion is infinite).

Lemma 1.4. *Let u_1 and v_1 be as in (1.1). Suppose there is a point z in $\mathbb{D}$ such that $|u_1(z)|^s + |v_1(z)|^{1-s} \leq \varepsilon$. Then the norm of the inclusion from $H^p(w^s)$ into $H^p + H^p(w)$ is at least $(1 - \varepsilon)/\varepsilon$.*

Proof. Suppose the norm of the embedding is C , so any f in $H^p(w^s)$ can be written as $g + h$, with

$$(1.5) \quad \|g\|_{H^p} + \|h\|_{H^p(w)} \leq C \|f\|_{H^p(w^s)}.$$

Writing

$$F = f \left(\frac{u_1}{v_1} \right)^s, \quad G = g, \quad H = h \left(\frac{u_1}{v_1} \right),$$

(1.5) is equivalent to saying that, for any F in H^p of norm one, there exist G, H in H^p , with $\|G\| + \|H\| \leq C$, and satisfying

$$\left(\frac{v_1}{u_1} \right)^s F = G + \left(\frac{v_1}{u_1} \right) H.$$

Now,

$$\begin{aligned} (1.6) \quad |G(z)| &\leq \exp \left[\int_{\mathbb{T}} \log |G(e^{i\theta})| P_z(\theta) d\sigma(\theta) \right] \\ &\leq \exp \left[\int_{\{\theta : |v_1(e^{i\theta})| < 1\}} \log(|v_1^s(e^{i\theta}) F(e^{i\theta})| + |v_1(e^{i\theta}) H(e^{i\theta})|) P_z(\theta) d\sigma(\theta) \right. \\ &\quad \left. + \int_{\{\theta : |v_1(e^{i\theta})| = 1\}} \log |G(e^{i\theta})| P_z(\theta) d\sigma(\theta) \right] \\ &\leq |v_1(z)|^s \exp \left[\int_{\mathbb{T}} \log(|F(e^{i\theta})| + |H(e^{i\theta})| + |G(e^{i\theta})|) P_z(\theta) d\sigma(\theta) \right]. \end{aligned}$$

So if $M(z, p)$ is the maximum value an H^p function of norm one can attain at z , (1.6) asserts that

$$|G(z)| \leq \|F\| + |G| + |H| \|M(z, p)|v_1(z)|^s \leq (C + 1)M(z, p)|v_1(z)|^s.$$

Similarly, $|H(z)| \leq (C + 1)M(z, p)|u_1(z)|^{1-s}$, so

$$(1.7) \quad |F(z)| \leq (C + 1)M(z, p)(|u_1(z)|^s + |v_1(z)|^{1-s}).$$

Letting F be that function that attains the value $M(z, p)$ at z , (1.7) and the hypothesis of the lemma imply that C is at least $(1 - \varepsilon)/\varepsilon$, as desired. $\square$

We can now prove that log-convex interpolation is possible if and only if $\log(w)$ is in BMO.

Theorem 1.8. *Let weights w_0 and w_1 be given, and let $0 < s < 1$. Then all linear operators T , continuous from $H^p(w_0)$ to $H^p(w_0)$ and from $H^p(w_1)$ to $H^p(w_1)$, extend to be continuous from $H^p(w_0^{1-s}w_1^s)$ to $H^p(w_0^{1-s}w_1^s)$, and satisfy $\|T\|_{H^p(w_0^{1-s}w_1^s)} \leq C\|T\|_{H^p(w_0)}^{1-s}\|T\|_{H^p(w_1)}^s$, for some constant C independent of T , if and only if $\log(w_1/w_0)$ is in BMO.*

Proof. Without loss of generality, assume $w_0 = 1$ and $w_1 = w$.

To prove sufficiency, we can show that $H^p(w^s)$ is isomorphic, under the hypothesis $\log(w) \in \text{BMO}$, to either the real interpolation space $K_{s,p}(H^p, H^p(w))$ or the complex interpolation space $[H^p, H^p(w)]_s$. Both these assertions are true, but as the complex space is a little simpler, we shall deal with it (see [Ca] or [BL] for a definition).

Because the complex method gives $L^p(w^s\sigma)$ when applied to L^p and $L^p(w\sigma)$, the complex interpolation space between H^p and $H^p(w\sigma)$ must be contained in $L^p(w\sigma)$, and hence, since all the elements in the interpolation space are also included in $H^p + H^p(w^s)$, they must be analytic (in particular in the Smirnov class) and so the interpolation space is always included in $H^p(w^s)$.

To get the reverse inclusion, letting ψ be the outer function with modulus $w^{-1/p}$, one must find, for any function $\psi^s f$ in $H^p(w^s)$, a function F , continuous on $\{\zeta \in \mathbb{C} : 0 \leq \Re(\zeta) \leq 1\}$ and analytic on the interior of the strip, that takes values in $H^p + H^p(w^s)$, on the line $\{\zeta \in \mathbb{C} : \Re(\zeta) = 0\}$ takes values in H^p , on the line $\{\zeta \in \mathbb{C} : \Re(\zeta) = 1\}$ takes values in $H^p(w)$, vanishes at infinity, and satisfies $F(s) = \psi^s f$ (the values of such functions F at s are, by definition, the elements of the space $[H^p, H^p(w)]_s$). The appropriate function is $F(\zeta) = e^{\zeta^2 - s^2} \psi^\zeta f$; this will work if $\Im(\log(\psi))$ is bounded, i.e., if $\log(w)$ has a bounded conjugate. Multiplying w by a weight that is bounded above and below will only affect constants, so this method will work if $\log(w)$ is the sum of a bounded function and one with bounded conjugate, which is equivalent to $\log(w)$ being in BMO [Ga, p. 248].

Necessity. Assume $\log(w)$ is not in BMO, and let u_t and v_t be as in (1.1). Then Lemmata 1.2 and 1.4 imply that, for any constant K , there exists a t so that the norm of the inclusion from $H^p((t^p w)^s)$ into $H^p + H^p(t^p w)$ is larger than K . By duality, there is a linear functional Λ , of norm at most one on both H^p and $H^p(t^p w)$, and norm at least K on $H^p((t^p w)^s)$. Now let f be any nonzero function in $H^p \cap H^p(w^s) \cap H^p(w)$ (this intersection is dense in all three spaces). Then the rank-one operator $\xi \mapsto \Lambda(\xi)f$ has norm at most $\|f\|_{H^p}$

on H^p , at most $t\|f\|_{H^p(w)}$ on $H^p(w)$, and at least $Kt^s\|f\|_{H^p(w^s)}$ on $H^p(w^s)$. Choosing K larger than $C\|f\|_{H^p}^{1-s}\|f\|_{H^p(w)}^s\|f\|_{H^p(w^s)}^{-1}$ yields a contradiction. $\square$

Remark. The theorem is also true for $0 < p < 1$. The necessity argument remains unchanged; for sufficiency, one can use the real method of interpolation (see [BL]). To get an upper bound on $K(t, f)$ one solves the corona problem for u_t and v_t to get functions U_t and V_t satisfying $u_t U_t + v_t V_t = 1$ and with $\|U_t\|$ and $\|V_t\|$ bounded independently of t (this can be done because of Lemma 1.2). Writing $f = u_t U_t f + v_t V_t f$ yields that the quasi-norms of f in $K_{s,p}(H^p, H^p(w))$ and $H^p(w^s)$ are equivalent.

When $\log(w)$ is not in BMO, the interpolation spaces $[H^p, H^p(w)]_s$ are still ones in which the polynomials are dense, evaluation at each point of the disk is continuous, and the shift is a cyclic contraction. Are the norms equivalent to ones realisable in some other way, e.g., as the $L^p(\mu)$ -norm for some measure μ defined on the disk? In general, when is a complex interpolation space between $P^2(\nu_0)$ and $P^2(\nu_1)$ naturally isomorphic to some $P^2(\mu)$ (where $P^2(\mu)$ denotes the closure of the polynomials in $L^2(\mu)$)?

Notice that Theorem 1.8 does not prove that, if $\log(w)$ fails to be in BMO, one cannot get $H^p(w^s)$ as an interpolation space between H^p and $H^p(w)$, only that one cannot get $H^p(w^s)$ as an interpolation space of exponent s . The method of proof of Theorem 1.8 will yield this stronger conclusion if one can pick the function f to have norm approximately 1 in H^p , $1/t$ in $H^p(w)$ and $1/t^s$ in $H^p(w^s)$. This in turn can be done if there is a set of positive measure where w is approximately t^{-p} , by choosing f to be a function with modulus one on this set, and modulus very small on the rest of the circle. For simplicity, we state this result for one special case:

Porism 1.9. *Suppose $\log(w)$ is not in BMO, and the supremum of the diameters of the bounded components of the complement of the essential range of $\log(w)$ is finite. Then there is a linear operator T , continuous from H^p to H^p and from $H^p(w)$ to $H^p(w)$, that fails to be continuous from $H^p(w^s)$ to $H^p(w^s)$.*

The hypotheses are satisfied, for example, by the weight $w(e^{i\theta}) = \theta$, $0 \leq \theta < 2\pi$.

We note that an indication that it may be possible to interpolate but not with log-convex bounds is given by considering the simple example of w_0 being 1 on the upper semicircle, ε on the lower semicircle; and w_1 being ε on the upper semicircle and 1 on the lower semicircle. It is easy to check that for any linear operator T , $\|T\|_{H^2(\sqrt{w_0 w_1})} \leq \sqrt{2} \max(\|T\|_{H^2(w_0)}, \|T\|_{H^2(w_1)})$. It is also easy to find operators with norms approximately one on both $H^2(w_0)$ and $H^2(\sqrt{w_0 w_1})$ and of order ε on $H^2(w_1)$. This behaviour makes the example in the next section easier to understand.

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We now give an example of a weight w whose logarithm is not in BMO, but for which $H^2(\sqrt{w})$ is an interpolation space between H^2 and $H^2(w)$.

Example 2.1. Parametrize the unit circle by θ going from $-\pi$ to π . For any point z in the disk, let ω_z denote harmonic measure on the circle with respect to z .

Define sets E_j and numbers α_j inductively as follows: Let $E_0 = \{e^{i\theta} : \pi \geq |\theta| \geq \pi/2\}$ and $\alpha_0 = \pi/2$. For each $j \geq 1$, choose $\alpha_j > 0$ so that the set $E_j := \{e^{i\theta} : \alpha_{j-1} > |\theta| \geq \alpha_j\}$ has the property that for each z in $\mathbb{D}$, either $\omega_z(\bigcup_{i=0}^{j-1} E_i) \leq 2^{-2^j}$ or $\omega_z(\mathbb{T} \setminus \bigcup_{i=0}^j E_i) \leq 2^{-2^j}$.

Let w be the weight that is 2^{2^j} on each E_j ; clearly $\log(w)$ is not in BMO. Moreover, by choosing α_j even smaller, if necessary, one can ensure that w is integrable (or in any desired L^p class).

For each j , let w_j be the weight $\min(w, 2^{2^j})$. Notice that

$$\sqrt{w} \leq \sum_{i=0}^{\infty} \frac{1}{2^{2^{i-1}}} w_i \leq 2\sqrt{w}.$$

So to prove that every linear operator T continuous from H^2 to H^2 and from $H^2(w)$ to $H^2(w)$ is also continuous from $H^2(\sqrt{w})$ to itself, it is enough to prove T is continuous from each $H^2(w_j)$ to itself, with a bound on the norm of T independent of j . And this in turn will follow if we can show that each function f of norm one in $H^2(w_j)$ can be written as $g + h$, with $\|g\|_{H^2} \leq C/2^{2^{j-1}}$ and $\|h\|_{H^2(w)} \leq C$ for some universal constant C .

Let t be $2^{-2^{j-1}}$, and let $u := u_t$ and $v := v_t$ be given by (1.1).

Claim. There is some $\delta > 0$, independent of j , such that

$$\inf\{|u(z)| + |v(z)| : z \in \mathbb{D}\} \geq \delta.$$

Assuming the claim for the moment, one can solve the corona problem for u and v to get functions U, V in H^∞ , with norms less than some constant C depending only on δ , such that $uU + vV = 1$. Writing $f = uUf + vVf$ then yields the desired decomposition.

To prove the claim, pick any point z in $\mathbb{D}$. Either $\omega_z(\bigcup_{i=0}^{j-1} E_i) \leq 2^{-2^j}$ or, for all $k \geq j$, $\omega_z(\bigcup_{i=0}^{k-1} E_i) \geq 2^{-2^k}$, so $\omega_z(E_k) \leq 2^{-2^{k-1}}$ for all $k \geq j+1$. In the former case,

$$\log |u(z)| = \sum_{i=0}^{j-1} \left[\frac{1}{2} \log(2)(2^i - 2^j) \omega_z(E_i) \right] > 2^{-2^j} (-j2^j),$$

and this is bounded below independently of j . In the latter case,

$$\log |v(z)| = \sum_{i=j+1}^{\infty} \left[\frac{1}{2} \log(2)(2^j - 2^i) \omega_z(E_i) \right] > - \sum_{i=j}^{\infty} 2^i 2^{-2^i},$$

and this too is bounded below. So the claim is proved. $\square$

Despite this behaviour, the proof of Theorem 1.8 can be sharpened to yield the following:

If $\log(w)$ is not in BMO, there exists some constant $C > 0$ such that, for any $\varepsilon > 0$, there exists a rank-one operator with norm less than C on both H^2 and $H^2(w)$, greater than 1 on $H^2(\sqrt{w})$, and less than ε on either H^2 or $H^2(w)$.

Suppose X_0 and X_1 are Banach spaces and X is another Banach space in which $X_0 \cap X_1$ is dense. Then the following proposition, whose proof is elementary, holds.

Proposition 3.1. *Let Y be a Banach space. Any linear map T that is continuous from X_0 to Y and from X_1 to Y is also continuous from X to Y if and only if X is continuously embedded in $X_0 + X_1$.*

Corollary 3.2. *All linear functionals continuous on $H^p(w_0)$ and $H^p(w_1)$ are also continuous on $H^p(w_0^{1-s}w_1^s)$ if and only if, for $w = w_1/w_0$, the functions u_1 and v_1 defined in (1.1) form a corona pair, i.e., satisfy $\inf_{z \in \mathbb{D}} |u_1(z)| + |v_1(z)| > 0$.*

Proof. Necessity follows from Lemma 1.4 and Proposition 3.1. To prove sufficiency, observe that u_1^s and v_1^{1-s} also form a corona pair, so there are H^∞ functions U and V satisfying $u_1^s(z)U(z) + v_1^{1-s}(z)V(z) = 1$. As in Example 2.1, any f in $H^p(w_0^{1-s}w_1^s)$ can be written as $Uu_1^s f + Vv_1^{1-s}f$, where the first term is in $H^p(w_0)$ and the second is in $H^p(w_1)$. $\square$

Clearly $L^p(\sqrt{w_0w_1}) \subseteq L^p(w_0) + L^p(w_1)$; however, this need not be true if L^p is replaced by H^p , as the following example shows.

Example 3.3. Let w_0 and w_1 be weights that satisfy

$$\lim_{\theta \downarrow \alpha} \frac{w_0(e^{i\theta})}{w_1(e^{i\theta})} = 0, \quad \lim_{\theta \uparrow \alpha} \frac{w_1(e^{i\theta})}{w_0(e^{i\theta})} = 0.$$

Then, for any $0 < s < 1$, $H^p(w_0^{1-s}w_1^s)$ is not contained in $H^p(w_0) + H^p(w_1)$.

Proof. By a theorem of Lindelöf [Ga, p. 92], if an H^∞ function has a one-sided limit of zero on the circle, it also has zero as a radial limit at that point; so both u_1 and v_1 will have radial limit zero at $e^{i\alpha}$ and cannot form a corona pair. $\square$

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Let $m(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic. The Toeplitz operator $T_{\overline{m}}$ is the linear operator whose action on monomials is given by $T_{\overline{m}}(z^k) = \overline{a_0}z^k + \overline{a_1}z^{k-1} + \cdots + \overline{a_k}z^0$. A problem that has been much studied recently is to determine, for a given weight w (or in general any measure on the circle), exactly which H^∞ functions m give rise to bounded operators $T_{\overline{m}}$ on $H^2(w)$ (see, e.g., [Sa, LS]).

A Helson-Szegö weight ρ is one that can be written as $e^{u+\bar{v}}$, where u is in L^∞ , and $\bar{v}$ is the conjugate of a function in L^∞ of norm less than $\pi/2$. In [DM^{CC}, Theorem 2.3], it was proved that if $w = |p|^2\rho$, where p is a polynomial and ρ is Helson-Szegö, then an H^∞ function m gives rise to a bounded operator $T_{\overline{m}}$ on $H^2(w)$ if and only if $m = T_{\overline{h}}f$, for some H^2 function f , where h is the outer function with modulus $w^{1/2}$ on the circle.

This theorem then solves the problem for all weights of the form $w(e^{i\theta}) = |e^{i\theta} - 1|^\alpha$, where $\alpha > -1$ is not an odd integer, because $|e^{i\theta} - 1|^\alpha$ is Helson-Szegö for $-1 < \alpha < 1$; using the affirmative part of Theorem 1.8, we can now almost solve the problem for α an odd integer:

Theorem 4.1. *Let $w(e^{i\theta}) = |e^{i\theta} - 1|^\alpha$, for any $\alpha > -1$. A necessary condition that the H^∞ -function m give rise to a bounded Toeplitz operator $T_{\overline{m}}$ on $H^2(w)$ is that $m = T_{(\overline{z-1})^{\alpha/2}}f$, for some f in H^2 ; a sufficient condition is that $m = T_{(\overline{z-1})^{(\alpha/2)+\varepsilon}}f$, for some $\varepsilon > 0$ and some f in H^2 .*

Proof. Necessity follows from [DM^{CC}, Proposition 2.2]; sufficiency follows from the theorem quoted above, and interpolating between the spaces

$H^2(|e^{i\theta} - 1|^{\alpha+2\varepsilon})$ and $H^2(|e^{i\theta} - 1|^{\alpha-2\varepsilon})$, which works because $\log(|e^{i\theta} - 1|)$ is in BMO. $\square$

How can Theorem 4.1 be improved to one in which the necessary and sufficient conditions coincide?

Toeplitz operators are a special subset of linear operators. For example, whatever weight w one chooses, if $T_{\overline{m}}$ is bounded on $H^2(w)$ and $H^2(\frac{1}{w})$, it will automatically be bounded on H^2 , with norm less than or equal to the minimum of the norms on the other two spaces. Can one say under what conditions the boundedness of a Toeplitz operator on $H^2(w_0)$ and on $H^2(w_1)$ implies its boundedness on $H^2(\sqrt{w_0 w_1})$?

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