NON-COHEN-MACAULAY SYMBOLIC BLOW-UPS FOR SPACE MONOMIAL CURVES

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ABSTRACT. Let $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$ denote the prime ideal in the formal power series ring $A = k[[X, Y, Z]]$ over a field k defining the space monomial curve $X = T^{n_1}$, $Y = T^{n_2}$, and $Z = T^{n_3}$ with $\text{GCD}(n_1, n_2, n_3) = 1$. Then the symbolic Rees algebra $R_s(\mathfrak{p}) = \bigoplus_{n \geq 0} \mathfrak{p}^{(n)}$ for $\mathfrak{p} = \mathfrak{p}(n^2 + 2n + 2, n^2 + 2n + 1, n^2 + n + 1)$ is Noetherian but not Cohen-Macaulay if $\text{ch } k = p > 0$ and $n = p^e$ with $e \geq 1$. The same is true for $\mathfrak{p} = \mathfrak{p}(n^2, n^2 + 1, n^2 + n + 1)$ if $\text{ch } k = p > 0$ and $n = p^e \geq 3$.

1. INTRODUCTION

Let A be a regular local ring of $\dim A = 3$ and $\mathfrak{p}$ a prime ideal of A with $\dim A/\mathfrak{p} = 1$. Let $R_s(\mathfrak{p}) = \sum_{n \geq 0} \mathfrak{p}^{(n)} t^n$ be the symbolic Rees algebra of $\mathfrak{p}$ (here t denotes an indeterminate over A). In this situation Goto, Nishida, and Shimoda [2] investigated the conditions under which the symbolic Rees algebra $R_s(\mathfrak{p})$ is a Gorenstein ring. They gave a criterion for the case in terms of the elements f and g of $\mathfrak{p}$ in Huneke's condition [5] of $R_s(\mathfrak{p})$ being Noetherian and showed that $R_s(\mathfrak{p})$ are necessarily Gorenstein for a certain class of prime ideals $\mathfrak{p}$ (cf. §2).

The problem when $R_s(\mathfrak{p})$ is Noetherian was raised by Cowsik [1] and is more fundamental. However there is an example $\mathfrak{p}$ due to Roberts [7] (passing to Nagata's counterexamples [6] to the 14th problem of Hilbert) whose symbolic Rees algebra $R_s(\mathfrak{p})$ is not Noetherian, while Cowsik's problem is still open for the general prime ideals $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$ in the formal power series ring $A = k[[X, Y, Z]]$ over a field k defining space monomial curves $X = T^{n_1}$, $Y = T^{n_2}$, and $Z = T^{n_3}$ with $\text{GCD}(n_1, n_2, n_3) = 1$.

Throughout the research [2] of Goto, Nishida, and Shimoda they seemed to follow a natural conjecture that $R_s(\mathfrak{p})$ has to be a Gorenstein ring once it is Noetherian for the space monomial curves $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$. We note here that $R_s(\mathfrak{p})$ is Gorenstein if it is Cohen-Macaulay (cf. [9, (3.4)]). Therefore the above conjecture is equivalent to saying that $R_s(\mathfrak{p})$ is always Cohen-Macaulay if it is Noetherian. Nevertheless this is not true in general, as we claim in the following

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Theorem 1.1. *Let p be a prime number and $n = p^e$ with $e \geq 1$. Then $R_s(\mathfrak{p})$ is Noetherian but not Cohen-Macaulay for $\mathfrak{p} = \mathfrak{p}(n^2 + 2n + 2, n^2 + 2n + 1, n^2 + n + 1)$ if $\text{ch } k = p$.*

If we choose $n = p = 2$ in Theorem 1.1, then we have $\mathfrak{p} = \mathfrak{p}(10, 9, 7)$, which was already cited in [2, (5.1)] too, while Vasconcelos [10, 4.1.4] gave the defining equations of $R_s(\mathfrak{p})$ for $\mathfrak{p} = \mathfrak{p}(10, 9, 7)$ claiming its Gorensteinness. The disparity between the authors and Vasconcelos is caused by his implicit assumption that $\text{ch } k = 0$ and quite interesting: *the Gorensteinness of symbolic Rees algebras depends on the characteristic of the ground field.*

Similarly as Theorem 1.1 we can prove that $R_s(\mathfrak{p})$ is Noetherian but not Cohen-Macaulay for $\mathfrak{p} = \mathfrak{p}(n^2, n^2 + 1, n^2 + n + 1)$ if $\text{ch } k = p > 0$ and if $n = p^e \geq 3$ (cf. §3). If we take $n = p = 3$, then $\mathfrak{p} = \mathfrak{p}(9, 10, 13)$. The symbolic Rees algebra $R_s(\mathfrak{p})$ for $\mathfrak{p} = \mathfrak{p}(9, 10, 13)$ remains non-Cohen-Macaulay when $\text{ch } k = 2$ (and when $\text{ch } k = 7$, too), which we will prove in the last section (cf. Theorem 4.1).

2. PRELIMINARIES

Let $(A, \mathfrak{m})$ be a regular local ring of $\dim A = 3$ and $\mathfrak{p}$ a prime ideal of A with $\dim A/\mathfrak{p} = 1$. First we shall recall Huneke's criterion for $R_s(\mathfrak{p})$ to be Noetherian.

Proposition 2.1 [5]. *If there exist $f \in \mathfrak{p}^{(k)}$ and $g \in \mathfrak{p}^{(l)}$ with positive integers k, l such that $l_A(A/(f, g, x)) = kl \cdot l_A(A/(x) + \mathfrak{p})$ for some $x \in \mathfrak{m} \setminus \mathfrak{p}$, then $R_s(\mathfrak{p})$ is Noetherian. When the field $A/\mathfrak{m}$ is infinite, the converse is also true. (Here $l_A(*)$ stands for the length.)*

With this criterion Huneke explored prime ideals $\mathfrak{p} = \mathfrak{p}(n_1, n_2, n_3)$ and proved that $R_s(\mathfrak{p})$ is Noetherian if $\min(n_1, n_2, n_3) = 4$.

In terms of the elements f and g the criterion of Goto, Nishida, and Shimoda is stated as follows, by which they showed $R_s(\mathfrak{p}(m, m+1, m+3))$ to be a Gorenstein ring for any $m \geq 1$.

Theorem 2.2 [2]. *Let f and g be as in Proposition 2.1. Then the following two conditions are equivalent.*

- (1) $R_s(\mathfrak{p})$ is a Gorenstein ring.
- (2) $A/(f, g) + \mathfrak{p}^{(n)}$ are Cohen-Macaulay for all $1 \leq n \leq k + l - 2$.

When this is the case, the A -algebra $R_s(\mathfrak{p})$ is generated by $\{\mathfrak{p}^{(n)}t^n\}_{1 \leq n \leq k+l-2}$, ft^k , and gt^l , and the rings $A/(f) + \mathfrak{p}^{(n)}$, $A/(g) + \mathfrak{p}^{(n)}$, and $A/(f, g) + \mathfrak{p}^{(n)}$ are Cohen-Macaulay for all $n \geq 1$.

Our proof of Theorem 1.1 consists of two steps. The first step is to find elements $f \in \mathfrak{p}$ and $g \in \mathfrak{p}^{(n+1)}$ satisfying Huneke's condition Proposition 2.1. The second one is to check that $A/(f) + \mathfrak{p}^{(2)}$ cannot be a Cohen-Macaulay ring for that choice of f . Then Theorem 2.2 will guarantee the assertion of Theorem 1.1. However for a large class of prime ideals $\mathfrak{p}$ the ring $A/(f) + \mathfrak{p}^{(2)}$ is always non-Cohen-Macaulay for any $f \in \mathfrak{p} \setminus \mathfrak{p}^{(2)}$, and it will make the second step obvious. So in what follows let us give a brief proof of this fact.

We now assume that the ideal $\mathfrak{p}$ is generated by the maximal minors of the

matrix

$$\begin{bmatrix} X^\alpha & Y^{\beta'} & Z^{\gamma'} \\ Y^\beta & Z^\gamma & X^{\alpha'} \end{bmatrix}$$

where X, Y, Z is a regular system of parameters for A and $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are positive integers. We furthermore assume that $\alpha > \alpha', \beta' > \beta$, and $\gamma' > \gamma$ (this condition is equivalent to saying that $\mathfrak{p}$ is *not* self-linked, cf. [3]). Let $a = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta'}$, $b = X^{\alpha+\alpha'} - Y^{\beta}Z^{\gamma'}$, and $c = Y^{\beta+\beta'} - X^{\alpha}Z^{\gamma}$ (hence $\mathfrak{p} = (a, b, c)$). Then because $X^\alpha a + Y^{\beta'}b + Z^{\gamma'}c = Y^\beta a + Z^\gamma b + X^{\alpha'}c = 0$, we have $(Y^\beta a + Z^\gamma b) \cdot X^{\alpha-\alpha'}a = -X^\alpha ac = (Y^{\beta'}b + Z^{\gamma'}c) \cdot c$, whence $Z^\gamma(-abX^{\alpha-\alpha'} + c^2Z^{\gamma'-\gamma}) = Y^\beta(a^2X^{\alpha-\alpha'} - Y^{\beta'-\beta}bc)$ so that $Y^\beta d_2 = -abX^{\alpha-\alpha'} + c^2Z^{\gamma'-\gamma}$ for some $d_2 \in \mathfrak{p}^{(2)}$. As $d_2 \equiv Y^{\beta+2\beta'}Z^{\gamma'-\gamma} \pmod{X}$, we get the equality

$$(X, d_2) + \mathfrak{p}^2 = (X, Y^{2(\beta+\beta')}, Y^{\beta+2\beta'}Z^{\gamma'-\gamma}, Y^{2\beta+\beta'}Z^{\gamma'}, \\ Y^{\beta+\beta'}Z^{\gamma+\gamma'}, Y^{2\beta}Z^{2\gamma'}, Y^\beta Z^{\gamma+2\gamma'}, Z^{2(\gamma+\gamma')},$$

which yields by an elementary computation that

$$l_A(A/(X, d_2) + \mathfrak{p}^2) = 3(\beta\gamma + \beta\gamma' + \beta'\gamma').$$

On the other hand, as X is a parameter for the Cohen-Macaulay local ring $A/\mathfrak{p}^{(2)}$, the length $l_A(A/(X) + \mathfrak{p}^{(2)})$ is equal to the multiplicity $e_{XA}(A/\mathfrak{p}^{(2)})$ of $A/\mathfrak{p}^{(2)}$ relative to X . Therefore because

$$l_A(A/(X) + \mathfrak{p}) = l_A(A/(X, Y^{\beta+\beta'}, Y^\beta Z^{\gamma'}, Z^{\gamma+\gamma'})), \\ = \beta\gamma + \beta\gamma' + \beta'\gamma',$$

we get by the additive formula [8, p. 126] of multiplicity that

$$l_A(A/(X) + \mathfrak{p}^{(2)}) = l_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}^2 A_{\mathfrak{p}}) \cdot l_A(A/(X) + \mathfrak{p}) \\ = 3(\beta\gamma + \beta\gamma' + \beta'\gamma').$$

Hence $l_A(A/(X) + \mathfrak{p}^{(2)}) = l_A(A/(X, d_2) + \mathfrak{p}^2)$ so that $(X) + \mathfrak{p}^{(2)} = (X, d_2) + \mathfrak{p}^2$. Thus $\mathfrak{p}^{(2)} = (d_2) + \mathfrak{p}^2 + X \cdot \mathfrak{p}^{(2)}$ and therefore Nakayama's lemma recovers the first assertion of the following lemma. As $\mu_A((X, d_2) + \mathfrak{p}^2) = 8$ and as $\mu_A(\mathfrak{p}^{(2)}) = \mu_A((X) + \mathfrak{p}^{(2)})/(X)$, the second assertion follows from the first. (Here $\mu_A(*)$ denotes the number of generators.)

Lemma 2.3 [4]. $\mathfrak{p}^{(2)} = (d_2) + \mathfrak{p}^2$ and $\mu_A(\mathfrak{p}^{(2)}) = 7$.

The key we need to prove Theorem 1.1 is the following

Proposition 2.4. $A/(f) + \mathfrak{p}^{(2)}$ is not a Cohen-Macaulay ring for any $f \in \mathfrak{p} \setminus \mathfrak{p}^{(2)}$.

Proof. Let $f \in \mathfrak{p} \setminus \mathfrak{p}^{(2)}$ and assume that $A/(f) + \mathfrak{p}^{(2)}$ is a Cohen-Macaulay ring. Then because the length $l_A(A/(X, f) + \mathfrak{p}^{(2)})$ is equal to the multiplicity $e_{XA}(A/(f) + \mathfrak{p}^{(2)})$ of $A/(f) + \mathfrak{p}^{(2)}$ relative to the parameter X , we get $l_A(A/(X, f) + \mathfrak{p}^{(2)}) = 2e$ by [8, p. 126] (here $e = l_A(A/(X) + \mathfrak{p})$). Let $B = A/(X)$, $I = (X) + \mathfrak{p}/(X)$, and $J = ((X) + \mathfrak{p}^{(2)})/(X)$. We put $C = B/J$. Then as $l_C(C/fC) = 2e$ and $l_C(C) = 3e$, we have $l_C(fC) = e$, whence $l_C((0):fC) = 2e$ so that $l_C(C/(0):fC) = e$. Consequently we must have $J:fB = I$, because $(0):fC \supseteq I/J$ (recall that $J \supseteq I^2$) and $l_C(C/(0):fC) = l_C(C/(I/J)) = e$.

Now let $y = Y \bmod(X)$ and $z = Z \bmod(X)$. Then by Lemma 2.3 we see $J = (y^{2(\beta+\beta')}, y^{\beta+2\beta'}z^{\gamma'-\gamma}, y^{2\beta+\beta'}z^{\gamma'}, y^{\beta+\beta'}z^{\gamma+\gamma'}, y^{2\beta}z^{2\gamma'}, y^{\beta}z^{\gamma+2\gamma'}, z^{2(\gamma+\gamma')})$, while $I = (y^{\beta+\beta'}, y^{\beta}z^{\gamma'}, z^{\gamma+\gamma'})$. Therefore a direct computation easily checks that $y^{\beta+\beta'-1}z^{\gamma'-1} \cdot I \subseteq J$ so that $y^{\beta+\beta'-1}z^{\gamma'-1} \in J: fB$, whence $y^{\beta+\beta'-1}z^{\gamma'-1} \in I = (y^{\beta+\beta'}, y^{\beta}z^{\gamma'}, z^{\gamma+\gamma'})$,

which is impossible. Thus $A/(f) + \mathfrak{p}^{(2)}$ cannot be Cohen-Macaulay.

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. First notice that the ideal $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^n & Y^n & Z^{n+1} \\ Y & Z & X \end{bmatrix}.$$

Let $a = Z^{n+2} - XY^n$, $b = X^{n+1} - YZ^{n+1}$, and $c = Y^{n+1} - X^nZ$. Then as $X^na + Y^nb + Z^{n+1}c = Ya + Zb + Xc = 0$, we have $Y^na^n + Z^nb^n + X^nc^n = 0$, whence

$$(Z^nb^n + X^nc^n)b = -Y^na^nb = (X^na + Z^{n+1}c)a^n.$$

Thus $X^n(a^{n+1} - bc^n) = Z^n(b^{n+1} - a^ncZ)$ and so $X^ne_{n+1} = b^{n+1} - a^ncZ$ for some $e_{n+1} \in \mathfrak{p}^{(n+1)}$. Because $c \equiv Y^{n+1} \bmod(Z)$ and $e_{n+1} \equiv X^{n^2+n+1} \bmod(Z)$, we get

$$l_A(A/(c, e_{n+1}, Z)) = 1 \cdot (n+1) \cdot l_A(A/(Z) + \mathfrak{p}) = (n+1)(n^2 + n + 1)$$

whence $R_s(\mathfrak{p})$ is, by Proposition 2.1, Noetherian. As $A/(c) + \mathfrak{p}^{(2)}$ is, by Proposition 2.4, not a Cohen-Macaulay ring, we conclude by Theorem 2.2 that $R_s(\mathfrak{p})$ is non-Cohen-Macaulay.

Corollary 3.1 [2]. $R_s(\mathfrak{p})$ is not a Cohen-Macaulay ring for $\mathfrak{p} = \mathfrak{p}(7, 9, 10)$ if $\text{ch } k = 2$.

Theorem 3.2. Let p be a prime number and $n = p^e$ with $e \geq 1$. Then $R_s(\mathfrak{p})$ is Noetherian but not Cohen-Macaulay for $\mathfrak{p} = \mathfrak{p}(n^2, n^2+1, n^2+n+1)$ if $\text{ch } k = p$ and $n \geq 3$.

Proof. The ideal $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^n & Y^n & Z^{n-1} \\ Y & Z & X \end{bmatrix}.$$

Let $a = Z^n - XY^n$, $b = X^{n+1} - YZ^{n-1}$, and $c = Y^{n+1} - X^nZ$. Then $X^na + Y^nb + Z^{n-1}c = Ya + Zb + Xc = 0$ so that $Y^na^n + Z^nb^n + X^nc^n = 0$, too. Similarly as in the proof of Theorem 1.1 we compare two expressions of Y^na^nb to get that $X^ne_{n+1} = b^{n+1}Z - a^nc$ for some $e_{n+1} \in \mathfrak{p}^{(n+1)}$. As $e_{n+1} \equiv (-1)^n Y^{n^2+n+1} \bmod(Z)$, we have

$$l_A(A/(b, e_{n+1}, Z)) = 1 \cdot (n+1) \cdot l_A(A/(Z) + \mathfrak{p}) = (n+1)(n^2 + n + 1),$$

whence $R_s(\mathfrak{p})$ is Noetherian by Proposition 2.1 but not Cohen-Macaulay by Theorem 2.2 and Proposition 2.4.

4. THE NON-COHEN-MACAULAYNESS OF $R_s(\mathfrak{p})$
FOR $\mathfrak{p} = \mathfrak{p}(9, 10, 13)$ IN THE CASE OF $\text{ch } k = 2$

Let $\mathfrak{p} = \mathfrak{p}(9, 10, 13)$. Then by Theorem 3.2 $R_s(\mathfrak{p})$ is Noetherian but not Cohen-Macaulay if $\text{ch } k = 3$. The same is true when $\text{ch } k = 2$ too, as we shall prove in the following

Theorem 4.1. *Suppose that $\text{ch } k = 2$. Then $R_s(\mathfrak{p})$ is Noetherian but non-Cohen-Macaulay for $\mathfrak{p} = \mathfrak{p}(9, 10, 13)$.*

We divide the proof of Theorem 4.1 into a few parts. First notice that $\mathfrak{p}$ is generated by the maximal minors of the matrix

$$\begin{bmatrix} X^3 & Y^3 & Z^2 \\ Y & Z & X \end{bmatrix}.$$

Let $a = Z^3 + XY^3$, $b = X^4 + YZ^2$, and $c = Y^4 + X^3Z$. Hence $\mathfrak{p} = (a, b, c)$. We put

$$(4.2) \quad d_2 = Y^7Z + X^2Z^5 + X^3Y^3Z^2 + X^7Y^2,$$

$$(4.3) \quad d_3 = Y^{10} + XZ^7 + X^2Y^3Z^4 + X^3Y^6Z + X^6Y^2Z^2 + X^{10}Y,$$

$$(4.4) \quad d_4 = Z^{11} + Y^{13}Z + X^3Y^9Z^2 + X^6Y^5Z^3 \\ + X^7Y^8 + X^9YZ^4 + X^{10}Y^4Z + X^{13}Z^2,$$

$$(4.5) \quad d_7 = Z^{19} + XY^3Z^{16} + X^3Y^{22} + X^4Y^{12}Z^7 + X^5Y^2Z^{14} + X^5Y^{15}Z^4 \\ + X^7Y^8Z^8 + X^8Y^{18}Z + X^9YZ^{12} + X^9Y^{14}Z^2 + X^{10}Y^4Z^9 \\ + X^{13}Y^{13} + X^{15}Y^6Z^4 + X^{18}Y^2Z^5 + X^{23}Y^4 + X^{26}Z.$$

Then a direct calculation checks the equalities

$$Xd_2 = b^2Y^2 + acZ, \quad Z^2d_3 = a^3X + bc^2Y, \\ Zd_4 = a^4 + bc^3, \quad Xd_7 = a^4d_3 + b^6c$$

so that

Lemma 4.6. $d_n \in \mathfrak{p}^{(n)}$ for $n = 2, 3, 4$, and 7 .

As $b^{10} + d_3d_7Z \equiv 0 \pmod{X}$, we have $Xd_{10} = b^{10} + d_3d_7Z$ with $d_{10} \in \mathfrak{p}^{(10)}$. Notice that $b \equiv YZ^2$, $d_3 \equiv Y^{10} + XZ^7$, and $d_7 \equiv Z^{19} + XY^3Z^{16} \pmod{X^2}$ and we get

$$(4.7) \quad d_{10} \equiv Z^{27} + Y^{13}Z^{17} \pmod{X}.$$

Thus $R_s(\mathfrak{p})$ is Noetherian by Proposition 2.1, because

$$l_A(A/(d_3, d_{10}, X)) = l_A(A/(X, Y^{10}, Z^{27})) = 270 = 3 \cdot 10 \cdot l_A(A/(X) + \mathfrak{p}).$$

To see that $R_s(\mathfrak{p})$ is not Cohen-Macaulay we need the following

Proposition 4.8. (1) $\mathfrak{p}^{(2)} = (d_2) + \mathfrak{p}^2$.

$$(2) \quad \mathfrak{p}^{(3)} = (d_3) + \mathfrak{p}^3.$$

$$(3) \quad \mathfrak{p}^{(4)} = (d_4) + d_3\mathfrak{p} + \mathfrak{p}^4.$$

$$(4) \quad \mathfrak{p}^{(5)} = \mathfrak{p}\mathfrak{p}^{(4)}.$$

$$(5) \quad \mathfrak{p}^{(6)} = (d_3^2) + \mathfrak{p}\mathfrak{p}^{(5)}.$$

- (6) $\mathfrak{p}^{(7)} = (d_7) + (d_3d_4) + \mathfrak{p}\mathfrak{p}^{(6)}.$
- (7) $\mathfrak{p}^{(8)} = \mathfrak{p}\mathfrak{p}^{(7)}.$
- (8) $\mathfrak{p}^{(9)} = (d_3^2) + d_7\mathfrak{p}^{(2)} + d_3d_4\mathfrak{p}^{(2)} + d_3^2\mathfrak{p}^3 + d_4\mathfrak{p}^5 + d_3\mathfrak{p}^6 + \mathfrak{p}^9.$
- (9) $\mathfrak{p}^{(10)} = (d_{10}, d_3d_7, d_3^2d_4) + \mathfrak{p}\mathfrak{p}^{(9)}.$
- (10) $\mathfrak{p}^{(11)} = \mathfrak{p}\mathfrak{p}^{(10)}.$

Proof. Let I_n ($2 \leq n \leq 11$) denote the ideals on the right-hand side of the above equalities. Then

$$l_A(A/(X) + I_n) \geq l_A(A/(X) + \mathfrak{p}^{(n)}) = e_{XA}(A/\mathfrak{p}^{(n)}) = 9 \cdot (n+1)n/2$$

for each $2 \leq n \leq 11$ (to see the last equality use the additive formula [8, p. 126] of multiplicity). Hence if we can show the inequality $l_A(A/(X) + I_n) \leq 9(n+1)n/2$, then we naturally have $(X) + I_n = (X) + \mathfrak{p}^{(n)}$ so that $\mathfrak{p}^{(n)} = I_n$ by Nakayama's lemma (cf. proof of Lemma 2.3). Checking of the inequalities $l_A(A/(X) + I_n) \leq 9(n+1)n/2$ is direct, which we leave to the readers.

Proposition 4.9. *The ring $A/(d_3) + \mathfrak{p}^{(n)}$ is Cohen-Macaulay for each $1 \leq n \leq 11$. Hence*

$$R_s(\mathfrak{p}) = A[at, bt, ct, d_2t^2, d_3t^3, d_4t^4, d_7t^7, d_{10}t^{10}].$$

Proof. The second assertion follows from the first (cf. [2, (3.7)(4)]). To check the first assertion we may assume $4 \leq n \leq 11$. It suffices to show

$$l_A(A/(X, d_3) + \mathfrak{p}^{(n)}) \leq e_{XA}(A/(d_3) + \mathfrak{p}^{(n)}).$$

Notice that $e_{XA}(A/(d_3) + \mathfrak{p}^{(n)}) = 9 \cdot l_{A_p}(A_p/d_3A_p + \mathfrak{p}^nA_p)$ by [8, p. 126], and we get

$$e_{XA}(A/(d_3) + \mathfrak{p}^{(n)}) = 27(n-1)$$

for $4 \leq n \leq 11$. Because we have already known by Proposition 4.8 the ideals $(X, d_3) + \mathfrak{p}^{(n)}$ explicitly, checking of the inequalities $l_A(A/(X, d_3) + \mathfrak{p}^{(n)}) \leq 27(n-1)$ for $4 \leq n \leq 11$ is elementary, whose detail is left to the readers.

Let us finish the proof of Theorem 4.1. By Theorem 2.2 it suffices to see that $A/(d_3, d_{10}) + \mathfrak{p}^{(11)}$ is not a Cohen-Macaulay ring. By Proposition 4.8 we have

$$(X, d_3, d_{10}) + \mathfrak{p}^{(11)} = (X) + (Y^{10}, Y^9Z^{24}, Y^7Z^{25}, Y^6Z^{26}, Z^{27})$$

so that $l_A(A/(X, d_3, d_{10}) + \mathfrak{p}^{(11)}) = 262$, while

$$e_{XA}(A/(d_3, d_{10}) + \mathfrak{p}^{(11)}) = 9 \cdot l_{A_p}(A_p/(d_3, d_{10})A_p + \mathfrak{p}^{11}A_p) = 9 \cdot 29 = 261$$

(recall that d_3, d_{10} forms a super regular sequence in A_p , cf. [2, (3.1)(3)]). Hence

$$l_A(A/(X, d_3, d_{10}) + \mathfrak{p}^{(11)}) > e_{XA}(A/(d_3, d_{10}) + \mathfrak{p}^{(11)})$$

and so $A/(d_3, d_{10}) + \mathfrak{p}^{(11)}$ cannot be Cohen-Macaulay. This completes the proof of Theorem 4.1.

Remark 4.10. Let M be the unique graded maximal ideal of $R_s(\mathfrak{p})$. Then we have

$$\dim R_s(\mathfrak{p})_M = 4 \quad \text{and} \quad \text{depth } R_s(\mathfrak{p})_M = 3$$

for the symbolic Rees algebras $R_s(\mathfrak{p})$ dealt with in Theorems 1.1, 3.2, and 4.1 (cf. [2, (2.1), (3.7)(3)]).

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