PROJECTIONS IN KAC-MOODY LIE ALGEBRAS

KAILASH C. MISRA AND MOHAN S. PUTCHA

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ABSTRACT. Let $\mathfrak{g}$ be a Kac-Moody Lie algebra and $\mathcal{B}$ its set of positive Borel subalgebras. If $\mathfrak{b} \in \mathcal{B}$ and $\mathfrak{p}$ is a parabolic subalgebra, let $\text{proj}_{\mathfrak{p}}(\mathfrak{b}) = \mathfrak{p} \cap \mathfrak{b} + \tau_{\mathfrak{n}}(\mathfrak{p})$ where $\tau_{\mathfrak{n}}(\mathfrak{p})$ denotes the nilradical of $\mathfrak{p}$. In this paper we consider the idempotent maps $E_{\mathfrak{p}, \mathfrak{p}^-} = \text{proj}_{\mathfrak{p}} \circ \text{proj}_{\mathfrak{p}^-} : \mathcal{B} \rightarrow \mathcal{B}$, where $\mathfrak{p}$ and $\mathfrak{p}^-$ are opposite parabolic subalgebras with $\mathfrak{p}$ being of positive type. We consider the semigroup $M = M(\mathfrak{g})$ generated (with respect to composition) by the maps $E_{\mathfrak{p}, \mathfrak{p}^-}$. In particular we show that the maximal subgroups of M are closely related to proper Levi subgroups of the Kac-Moody group associated with $\mathfrak{g}$.

INTRODUCTION

Projections in algebraic groups were introduced and used by Tits [6] in his theory of buildings. If G is a simple algebraic group, B a Borel subgroup of G , and P a parabolic subgroup of G , then $\text{proj}_P(B) = (P \cap B)R_u(P)$ where $R_u(P)$ denotes the unipotent radical of P . In this way, we have an idempotent map $\text{proj}_P : G/B \rightarrow G/B$. One of the authors [5] studied the monoid M (with respect to composition) generated by these projections. In particular, it was shown in [5] that the maximal subgroups of M are closely related to proper Levi subgroups of G .

In this paper we study projections in a Kac-Moody Lie algebra. Let $\mathfrak{g}$ be a Kac-Moody Lie algebra associated to an indecomposable symmetrizable generalized Cartan matrix and let G be the corresponding Kac-Moody group [2, 3, 4]. Let $\mathfrak{b}_+$ and $\mathfrak{b}_-$ denote the standard positive and negative Borel subalgebras of $\mathfrak{g}$. It was proved in [4] that any Borel subalgebra of $\mathfrak{g}$ is $\text{Ad } G$ -conjugate to $\mathfrak{b}_+$ or $\mathfrak{b}_-$. We say a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is of positive type if it is conjugate to $\mathfrak{b}_+$ and that a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is of positive type if it contains a Borel subalgebra of positive type. Let $\mathcal{B}$ (resp. $\mathcal{P}$) denote the set of all Borel (resp. parabolic) subalgebras of $\mathfrak{g}$ of positive type. The parabolic subalgebra $\mathfrak{p}^-$ is said to be an opposite of the parabolic subalgebra $\mathfrak{p}$ if $\mathfrak{p} \cap \mathfrak{p}^-$ is a common Levi factor. For $\mathfrak{b} \in \mathcal{B}$ and $\mathfrak{p}$ any parabolic subalgebra, we define $\text{proj}_{\mathfrak{p}}(\mathfrak{b}) = \mathfrak{p} \cap \mathfrak{b} + \tau_{\mathfrak{n}}(\mathfrak{p})$, where $\tau_{\mathfrak{n}}(\mathfrak{p})$ denotes the nilradical of $\mathfrak{p}$. In general $\text{proj}_{\mathfrak{p}}(\mathfrak{b})$ does not belong to $\mathcal{B}$ (unless $\mathfrak{p} \in \mathcal{P}$). We consider the idempotent maps $E_{\mathfrak{p}, \mathfrak{p}^-} : \mathcal{B} \rightarrow \mathcal{B}$, where $E_{\mathfrak{p}, \mathfrak{p}^-} = \text{proj}_{\mathfrak{p}} \circ \text{proj}_{\mathfrak{p}^-}$ and $\mathfrak{p} \in \mathcal{P}$, $\mathfrak{p}^-$

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an opposite of $\mathfrak{p}$. Let $M = M(\mathfrak{g})$ be the monoid generated by $E_{\mathfrak{p}, \mathfrak{p}^-}$, $\mathfrak{p} \in \mathcal{P}$, with respect to composition. Let $\mathfrak{l} = \mathfrak{p} \cap \mathfrak{p}^-$, $\text{St}_G(\mathfrak{l}) = \{x \in G \mid \text{Ad } x(\mathfrak{l}) = \mathfrak{l}\}$, $H = \{x \in \text{St}_G(\mathfrak{l}) \mid \text{Ad } x(\mathfrak{b}) = \mathfrak{b}, \text{ for all } \mathfrak{b} \in \mathcal{B}(\mathfrak{l})\}$, and $\tilde{L} = \text{St}_G(\mathfrak{l})/H$. In this paper it is shown (Theorem 2.8) that the maximal subgroup of M with identity element $E_{\mathfrak{p}, \mathfrak{p}^-}$ is isomorphic to $\tilde{L}$. Since the Dynkin diagram of any Kac-Moody Lie algebra can be extended by a single node, we see that a group closely related to any Kac-Moody group arises as a maximal subgroup of some M .

1. PRELIMINARIES

In this section we will recall some basic definitions and facts about Kac-Moody algebras and groups associated to a symmetrizable generalized Cartan matrix. For more details the reader is referred to [2, 3, 4].

Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ symmetrizable generalized Cartan matrix, i.e., an integral matrix satisfying $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$, and DA is symmetric for some nondegenerated diagonal matrix D . We assume for simplicity that A is indecomposable. Let $\mathbb{C}$ denote the field of complex numbers and $\mathbb{C}^*$ denote the set of nonzero complex numbers. Consider the Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ over $\mathbb{C}$ (see [2]) with generators e_i, f_i, h_i , $i \in I = \{1, 2, \dots, n\}$.

The Lie algebra $\mathfrak{g}$ admits a gradation $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_{\alpha}$ by the free abelian group Q on symbols α_i , $i \in I$, which is called the root lattice, such that $\mathfrak{h} = \mathfrak{g}_0 = \bigoplus_{i \in I} \mathbb{C}h_i$, $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$, and $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$. Call $\mathfrak{h}$ the standard Cartan subalgebra of $\mathfrak{g}$. Let $\Delta = \{\alpha \in Q \mid \mathfrak{g}_{\alpha} \neq 0, \alpha \neq 0\}$ be the set of roots of $\mathfrak{g}(A)$. Define $r_i \in \text{Aut}_{\mathbb{C}}(\mathfrak{h})$, $i \in I$ by $r_i(h) = h - \alpha_i(h)h_i$ and take $S = \{r_i \mid i \in I\}$. S generates the Weyl group W , and (W, S) is a Coxeter system. Let Γ be the Coxeter graph for W . Since A is indecomposable, Γ is connected. W preserves the root system Δ . A real (resp. imaginary) root is an element of $\Delta^{\text{re}} := \{w \cdot \alpha_i \mid w \in W, i \in I\}$ (resp. $\Delta^{\text{im}} := \Delta \setminus \Delta^{\text{re}}$). Elements of $\Delta_+ := Q_+ \cap \Delta$ are called positive roots and $\Delta_- := \Delta \setminus \Delta_+$ are called negative roots. Define $\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \Delta_{\pm}} \mathfrak{g}_{\alpha}$. Then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}_{\pm} = \mathfrak{h} \oplus \mathfrak{n}^{\pm}$ are the standard Borel subalgebras (positive if '+', negative if '-').

For $t \in \mathbb{C}^*$ and $u \in \mathbb{C}$, define the following elements of $\text{SL}_2(\mathbb{C})$:

$$h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad x(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad y(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

The following axioms (1.1)–(1.3) determine (uniquely up to isomorphism) a group $G = G(A)$ and homomorphisms $\phi_i: \text{SL}_2(\mathbb{C}) \rightarrow G(A)$ for $i \in I$ [3]. Here and further on, $\phi_i(h(t))$, $\phi_i(x(u))$, and $\phi_i(y(u))$ are denoted by $h_i(t)$, $x_i(u)$, and $y_i(u)$, respectively, for short.

(1.1) There exists a faithful G -module (V, π) over $\mathbb{C}$ such that each $\text{SL}_2(\mathbb{C})$ -module $(V, \pi \circ \phi_i)$ is a direct sum of rational finite-dimensional submodules.

(1.2) $h_i(t)x_j(u)h_i(t)^{-1} = x_j(t^{a_{ij}}u)$ and $h_i(t)y_j(u)h_i(t)^{-1} = y_j(t^{-a_{ij}}u)$ for all $i, j \in I$, $t \in \mathbb{C}^*$, and $u \in \mathbb{C}$.

(1.3) If a group G' and homomorphisms $\phi'_i: \text{SL}_2(\mathbb{C}) \rightarrow G'$, $i \in I$, satisfy (1.2) and (1.3), then there exists a unique homomorphism $\psi: G \rightarrow G'$ such that $\phi'_i = \psi \circ \phi_i$ for all $i \in I$.

Put $G_i = \phi_i(\text{SL}_2(\mathbb{C}))$, $i \in I$. It follows from the axioms that the subgroups

G_i , $i \in I$, generate the group G . Put $H_i = \{h_i(t) | t \in \mathbb{C}^*\}$, and let H be the subgroup of G generated by the subgroups H_i . Then H is an abelian subgroup of G . Let N_i be the normalizer of H_i in G_i and let N be the subgroup of G generated by the N_i 's. Then H is an abelian normal subgroup of N and the Weyl group W is isomorphic to the quotient group N/H [3, 4].

To the integrable $\mathfrak{g}$ -module $(\mathfrak{g}, \text{ad})$ associate the homomorphism $\text{Ad}: G \rightarrow \text{Aut}_{\mathbb{C}}(\mathfrak{g})$ satisfying $\text{Ad}(\exp x) = \exp(\text{ad } x)$ for all $x \in \mathfrak{g}_{\alpha}$, $\alpha \in \Delta^{\text{re}}$. Then the kernel of Ad is the center $C(G)$ of G . For $i \in I$ consider the one-parameter subgroups $U_{\alpha_i} = \{x_i(u) | u \in \mathbb{C}\}$, $U_{-\alpha_i} = \{y_i(u) | u \in \mathbb{C}\}$ of G . For a real root $\alpha = w\alpha_i$, take $n \in N$ such that $w = nH$ and put $U_{\pm\alpha} = nU_{\pm\alpha_i}n^{-1}$. Then $U_{\alpha} = \exp \mathfrak{g}_{\alpha}$ for all $\alpha \in \Delta^{\text{re}}$ [3]. Let U_+ (resp. U_-) be the subgroup of G generated by the subgroups U_{α} (resp. $U_{-\alpha}$), $\alpha \in \Delta_+^{\text{re}}$. Define the standard Borel subgroups $B_{\pm} = HU_{\pm}$. Then we have [3, 4]

$$(1.4) \quad G = \bigcup_{w \in W} B_+ w B_+ \quad (\text{Bruhat decomposition}),$$

$$(1.5) \quad G = \bigcup_{w \in W} B_- w B_+ \quad (\text{Birkhoff decomposition}).$$

The following theorem is known [4, Theorems 2, 3].

Theorem 1.1 [4]. (i) Every Cartan subalgebra of $\mathfrak{g}$ is $\text{Ad}(G)$ -conjugate to $\mathfrak{h}$.

(ii) Every Borel subalgebra of $\mathfrak{g}$ is $\text{Ad}(G)$ -conjugate to $\mathfrak{b}_+$ or to $\mathfrak{b}_-$.

(iii) Every Cartan subalgebra of $\mathfrak{b}_+$ is $\text{Ad}(U_+)$ -conjugate to $\mathfrak{h}$.

The following is an immediate corollary of Theorem 1.1(ii) and equations (1.4), (1.5).

Corollary 1.2. If $\mathfrak{p}_1, \mathfrak{p}_2$ are any two parabolic subalgebras of $\mathfrak{g}$, then $\mathfrak{p}_1 \cap \mathfrak{p}_2$ contains a Cartan subalgebra of $\mathfrak{g}$. In particular, if $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are any two Borel subalgebras of $\mathfrak{g}$, then $\mathfrak{b}_1 \cap \mathfrak{b}_2$ contains a Cartan subalgebra of $\mathfrak{g}$.

Let $\mathfrak{b}$ be any Borel subalgebra and $\mathfrak{p}$ any parabolic subalgebra of $\mathfrak{g}$. Define $B = \{g \in G | (\text{Ad } g)(\mathfrak{b}) = \mathfrak{b}\}$ and $P = \{g \in G | (\text{Ad } g)(\mathfrak{p}) = \mathfrak{p}\}$. Then B is a Borel subgroup and P is a parabolic subgroup of G . We define B (resp. P) to be the Borel (resp. parabolic) subgroup associated with the Borel (resp. parabolic) subalgebra $\mathfrak{b}$ (resp. $\mathfrak{p}$) of $\mathfrak{g}$. A parabolic subgroup P^- is said to be an *opposite* of the parabolic subgroup P of G if $L = P \cap P^-$ is a reductive group. Then we have a Levi decomposition $P = LU$ where $U = R_u(P)$ is the unipotent radical of P . For any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{c}$, the Levi factor $\mathfrak{l}$ of $\mathfrak{p}$ is the subalgebra spanned by $\mathfrak{c}$ and all pairs of opposite root spaces with respect to $\mathfrak{c}$ occurring in $\mathfrak{p}$. Then we have the Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{u} = \mathfrak{r}_n(\mathfrak{p})$ is the nilradical of $\mathfrak{p}$. A parabolic subalgebra $\mathfrak{p}^-$ is said to be an *opposite* of the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ if $\mathfrak{l} = \mathfrak{p} \cap \mathfrak{p}^-$ is a Levi factor of $\mathfrak{p}$ (and $\mathfrak{p}^-$).

2. MAIN SECTION

As in §1, let $\mathfrak{g}$ and G denote the Kac-Moody algebra and Kac-Moody group, respectively, associated to a symmetrizable indecomposable Cartan matrix A . We say a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is of positive (resp. negative) type if $\mathfrak{b}$ is $\text{Ad } G$ -conjugate to the standard positive (resp. negative) Borel subalgebra $\mathfrak{b}_+$

(resp. b_-). A parabolic subalgebra p of g is of positive (resp. negative) type if p contains a Borel subalgebra of positive (resp. negative) type. Let $\mathcal{B} = \mathcal{B}(g)$ denote the set of Borel subalgebras of g of positive type and $\mathcal{P} = \mathcal{P}(g)$ denote the set of parabolic subalgebras of g of positive type. For any subalgebra $\mathfrak{k}$ of g , whenever defined we will write $\mathcal{B}(\mathfrak{k})$ and $\mathcal{P}(\mathfrak{k})$ to denote the set of Borel subalgebras and parabolic subalgebras, respectively, of positive type.

For any Borel subalgebra b and parabolic subalgebra p of g define

$$(2.1) \quad \text{proj}_p(b) = p \cap b + \tau_n(p),$$

where $\tau_n(p)$ is the nilradical of p . For $b \in \mathcal{B}$ and $p \in \mathcal{P}$ define

$$(2.2) \quad E_{p,p^-} = \text{proj}_p \circ \text{proj}_{p^-},$$

where p^- is an opposite of p . Then $E_{p,p^-}(b) \in \mathcal{B}$ for all $b \in \mathcal{B}$, $E_{g,g} = 1$, and $E_{p,p^-} \circ E_{p,p^-} = E_{p,p^-}$. Let $\mathcal{E}$ denote the set of idempotent maps E_{p,p^-} , $p \in \mathcal{P}$. Observe that if $l = p \cap p^-$ and $b \in \mathcal{B}$ have a common Cartan subalgebra then $E_{p,p^-}(b) = \text{proj}_p(b)$. For a subalgebra $\mathfrak{k} \subseteq g$ and $x \in G$, we will write ${}^x\mathfrak{k}$ to denote $(\text{Ad } x)(\mathfrak{k})$ and $\mathfrak{k}^x$ to denote $(\text{Ad } x^{-1})(\mathfrak{k})$. Then the next lemma follows immediately from the definitions.

Lemma 2.1. *For $x \in G$ and $p \in \mathcal{P}$ we have (i) $\text{Ad } x \circ E_{p,p^-} = E_{x p, x p^-} \circ \text{Ad } x$, and (ii) $E_{p,p^-} \circ \text{Ad } x = \text{Ad } x \circ E_{p^x, p^{-x}}$.*

For two parabolic subalgebras p and q of g , we define $p \leftrightarrow q$ if and only if p and q have a common Levi factor. We say $E_{p,p^-} \leftrightarrow E_{q,q^-}$ if $p^- \leftrightarrow q$.

Lemma 2.2. *Let $p, q \in \mathcal{P}$ with respective opposites p^- and q^- . Suppose $p \cap p^- \cap q \cap q^-$ contains a Cartan subalgebra of g . Then there exists $p_1, q_1 \in \mathcal{P}$ with respective opposites p_1^- and q_1^- such that $p_1 \subseteq p$, $p_1^- \subseteq p^-$, $q_1 \subseteq q$, $q_1^- \subseteq q^-$, $p_1^- \leftrightarrow q_1$ (so $E_{p_1, p_1^-} \leftrightarrow E_{q_1, q_1^-}$), and $E_{p,p^-} \circ E_{q,q^-} = E_{p_1, p_1^-} \circ E_{q_1, q_1^-}$.*

Proof. Let $l_1 = p \cap p^-$, $l_2 = q \cap q^-$, and $l = l_1 \cap l_2$. Then we have the Levi decompositions $p = l_1 + u$, $p^- = l_1 + u^-$, $q = l_2 + v$, $q^- = l_2 + v^-$, where $u = \tau_n(p)$, $u^- = \tau_n(p^-)$, $v = \tau_n(q)$, $v^- = \tau_n(q^-)$. Let P, P^-, Q, Q^- denote the parabolic subgroups associated with the parabolic subalgebras p, p^-, q, q^- , respectively. We have the Levi decomposition of the parabolic subgroups $P = L_1 U$, $P^- = L_1 U^-$, $Q = L_2 V$, and $Q^- = L_2 V^-$, where $U = R_u(P)$, $U^- = R_u(P^-)$, $V = R_u(Q)$, and $V^- = R_u(Q^-)$. Consider the parabolic subalgebras

$$\begin{aligned} p_1 &= (p \cap q) + u = l + (l_1 \cap v) + u, \\ p_1^- &= (p^- \cap q^-) + u^- = l + (l_1 \cap v^-) + u^-, \\ q_1 &= (p \cap q) + v = l + (l_2 \cap u) + v, \\ q_1^- &= (p^- \cap q^-) + v^- = l + (l_2 \cap u^-) + v^-. \end{aligned}$$

Observe that p_1^- and q_1 have l as common Levi factor. Let $b \in \mathcal{B}$. By Corollary 1.2, $b \cap q_1$ contains a Cartan subalgebra h , say. There exists $x \in L_2 \cap U^-$, $y \in V^-$ such that ${}^{xy}h \subseteq l$. Therefore, ${}^y h \subseteq l_2$. So $E_{q,q^-}(b) = (l_2 \cap {}^y b) + v$. Also ${}^{xy}h \subseteq l$, $x \in L_2 \cap U^-$ implies that ${}^y h \subseteq p^-$. Hence

$$E_{p,p^-} \circ E_{q,q^-}(b) = (l \cap {}^{xy}b) + (l_1 \cap {}^x v) + u = (l \cap {}^{xy}b) + (l_1 \cap v) + u.$$

Now $v = {}^x v$ as $x \in L_2 \subseteq Q$. But

$$E_{q_1, q_1^-}(b) = (l \cap {}^{xy}b) + (l_2 \cap u) + v,$$

so

$$E_{p_1, p_1^-} \circ E_{q_1, q_1^-}(b) = (l \cap {}^{xy}b) + (l_1 \cap v) + u = E_{p, p^-} \circ E_{q, q^-}(b),$$

which proves the lemma. $\square$

Corollary 2.3. *Let $p, q \in \mathcal{P}$ with respective opposites p^- and q^- . Then there exists $p_1, q_1 \in \mathcal{P}$ with respective opposites p_1^- and q_1^- such that $p_1^- \leftrightarrow q_1$ (so $E_{p_1, p_1^-} \leftrightarrow E_{q_1, q_1^-}$), $p_1 \subseteq p$, $p_1^- \subseteq p^-$, $q_1 \subseteq q$, $q_1^- \subseteq q^-$, and $E_{p, p^-} \circ E_{q, q^-} = E_{p_1, p_1^-} \circ E_{q_1, q_1^-}$.*

Proof. Let $l_1 = p \cap p^-$, $l_2 = q \cap q^-$. By Corollary 1.2, $p^- \cap q$ contains a Cartan subalgebra $\mathfrak{h}$, say. Let P, P^-, Q, Q^- denote the parabolic subgroups associated with p, p^-, q , and q^- , respectively. Let $U = R_u(P)$, $U^- = R_u(P^-)$, $V = R_u(Q)$, and $V^- = R_u(Q^-)$. Then there exists $x \in U^-$ and $y \in V$ such that $\mathfrak{h} \subseteq l_1^x$ and $\mathfrak{h} \subseteq l_2^y$. Then using Lemma 2.1,

$$E_{p, p^-} \circ E_{q, q^-} = E_{p, p^-} \circ \text{Ad } x \circ \text{Ad } y \circ E_{q, q^-} = \text{Ad } x \circ E_{p^x, p^-} \circ E_{q, q^-} \circ \text{Ad } y$$

since $p^{-x} = p^-$ and ${}^y q = q$. By Lemma 2.2, since $\mathfrak{h} \subseteq p^x \cap p^- \cap q \cap {}^y q^-$, we can find $p_1, q_1 \in \mathcal{P}$ with opposites p_1^- , q_1^- , respectively, such that $p_1 \subseteq p^x$, $p_1^- \subseteq p^-$, $q_1 \subseteq q$, $q_1^- \subseteq {}^y q^-$, $p_1^- \leftrightarrow q_1$, and $E_{p^x, p^-} \circ E_{q, {}^y q^-} = E_{p_1, p_1^-} \circ E_{q_1, q_1^-}$. Hence using Lemma 2.1,

$$\begin{aligned} E_{p, p^-} \circ E_{q, q^-} &= \text{Ad } x \circ E_{p_1, p_1^-} \circ E_{q_1, q_1^-} \circ \text{Ad } y \\ &= E_{x p_1, p_1^-} \circ \text{Ad } x \circ \text{Ad } y \circ E_{q_1, q_1^{-y}} = E_{x p_1, p_1^-} \circ E_{q_1, q_1^{-y}}, \end{aligned}$$

since $x \in U^-$, $y \in V$, so that ${}^x p_1^- = p_1^-$ and $q_1^y = q_1$. Hence the corollary follows. $\square$

Corollary 2.4. *Let $p, q \in \mathcal{P}$ with respective opposites p^- and q^- . Suppose $p \subseteq q$ and $p^- \subseteq q^-$. Then $E_{p, p^-} \circ E_{q, q^-} = E_{p, p^-} = E_{q, q^-} \circ E_{p, p^-}$.*

Proof. By Lemma 2.2, we have

$$E_{p, p^-} \circ E_{q, q^-} = E_{p_1, p_1^-} \circ E_{q_1, q_1^-}$$

where $p_1 = p \cap q + \tau_n(p) = p$, $p_1^- = p^- \cap q^- + \tau_n(p^-) = p^-$, $q_1 = p \cap q + \tau_n(q) = p$, and $q_1^- = p^- \cap q^- + \tau_n(q^-) = p^-$. Hence $E_{p, p^-} \circ E_{q, q^-} = E_{p, p^-} \circ E_{p, p^-} = E_{p, p^-}$. Similarly, $E_{q, q^-} \circ E_{p, p^-} = E_{p, p^-}$. $\square$

Let M denote the monoid generated by the idempotent maps $E_{p, p^-} \in \mathcal{E}$ with respect to composition.

Proposition 2.5. *Let $p \in \mathcal{P}$ be a maximal parabolic subalgebra of $\mathfrak{g}$ with opposite p^- . Let P and P^- be the parabolic subgroups of G associated with p and p^- , respectively, and $L = P \cap P^-$. Let $\mathfrak{A}_1 = \{x \in G \mid E_{p, p^-} \circ \text{Ad } x \circ E_{p, p^-} \in M\}$. Then $L \subseteq \mathfrak{A}_1$.*

Proof. Without loss of generality, we may assume that $P = P_J$ for some $J = I \setminus \{i\}$, $1 \leq i \leq n$. Let $L = L_J$, $P^- = P_J^-$, $P = LU$, where $U = U_J = R_u(P_J)$ and $P^- = LU^-$, where $U^- = U_J^- = R_u(P_J^-)$. Let $u \in U$, $v \in U^-$. Then by

Lemma 2.1, we have

$$\begin{aligned} E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } u \circ \text{Ad } v \circ E_{\mathfrak{p}, \mathfrak{p}^-} \\ = \text{Ad } u \circ E_{\mathfrak{p}, \mathfrak{p}^-} \circ E_{v, \mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } v \quad \text{since } \mathfrak{p}^u = \mathfrak{p}, \quad v\mathfrak{p}^- = \mathfrak{p}_1^- \\ = E_{\mathfrak{p}, \mathfrak{p}^-} \circ E_{v, \mathfrak{p}, \mathfrak{p}^-} \quad \text{since } u \in U, \quad v \in U^-. \end{aligned}$$

Hence $UU^- \subseteq \mathfrak{A}_1$. Again, since $E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } v = E_{\mathfrak{p}, \mathfrak{p}^-}$ as $v \in U^-$ and $\text{Ad } u \circ E_{\mathfrak{p}, \mathfrak{p}^-} = E_{\mathfrak{p}, \mathfrak{p}^-}$ as $u \in U$, it follows that $U^-UU^-U \subseteq \mathfrak{A}_1$. Let $\sigma = \alpha_i$ and $U_\sigma, U_{-\sigma}$ be the two root subgroups corresponding to σ and $-\sigma$, respectively. Consider the group $G_\sigma = \langle U_\sigma, U_{-\sigma} \rangle$ that is isomorphic to $\text{SL}(2, \mathbb{C})$. One then observes by direct computation that $G_\sigma = U_{-\sigma}U_\sigma U_{-\sigma}U_\sigma$. Since $\sigma \notin \{\alpha_j | j \in J\}$ (see [1]) we have $U_\sigma \subseteq U$ and $U_{-\sigma} \subseteq U^-$. Thus $G_\sigma \subseteq U^-UU^-U \subseteq \mathfrak{A}_1$. Hence the maximal torus T_σ of G_σ is contained in $\mathfrak{A}_1$.

Now let σ_1 and σ_2 be two simple roots corresponding to two adjacent nodes in the Coxeter graph Γ of G and suppose that $G_{\sigma_1} \subseteq \mathfrak{A}_1$. Then $T_{\sigma_1} \subseteq L_{\sigma_2} \cap \mathfrak{A}_1 \triangleleft L_{\sigma_2} = C(L_{\sigma_2})G_{\sigma_2}$. But since σ_1 and σ_2 are adjacent, $T_{\sigma_1} \not\subseteq C(L_{\sigma_2})$. Hence $G_{\sigma_2} \subseteq L_{\sigma_2} \cap \mathfrak{A}_1$, so $G_{\sigma_2} \subseteq \mathfrak{A}_1$. Clearly, $C(L) \subseteq \mathfrak{A}_1$. Since the generalized Cartan matrix $A = (a_{ij})$ is indecomposable, Γ is connected. Hence $G_\sigma \subseteq \mathfrak{A}_1$ for all $\sigma \in \{\alpha_i | i \in I\}$. Since L is generated by $C(L)$ and G_σ , $\sigma \in \{\alpha_i | i \in J\}$, it follows that $L \subseteq \mathfrak{A}_1$. $\square$

Proposition 2.6. Let $\mathfrak{p} \in \mathcal{P}$, $\mathfrak{p} \neq \mathfrak{g}$, with opposite $\mathfrak{p}^-$. Let $\mathfrak{A} = \{x \in G | E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } x \in M\}$. Then $\mathfrak{A} = G$.

Proof. It is enough to prove the result for $\mathfrak{p}$ maximal. For if $\mathfrak{p}_1 \in \mathcal{P}$, $\mathfrak{p}_1 \subseteq \mathfrak{p}$, $\mathfrak{p}_1^- \subseteq \mathfrak{p}^-$, then by Corollary 2.4,

$$E_{\mathfrak{p}_1, \mathfrak{p}_1^-} \circ \text{Ad } x = E_{\mathfrak{p}_1, \mathfrak{p}_1^-} \circ (E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } x) \in M$$

if $E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } x \in M$. Let P, P^- denote the parabolic subgroups associated with $\mathfrak{p}$ and $\mathfrak{p}^-$, respectively. Let $L = P \cap P^-$, $P = LU$, $P^- = LU^-$, where $U = R_u(P)$ and $U^- = R_u(P^-)$. Let $u \in U$, $v \in U^-$. Then by Lemma 2.1,

$$\begin{aligned} E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } u \circ \text{Ad } v &= E_{\mathfrak{p}, \mathfrak{p}^-} \circ E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } u \circ \text{Ad } v \\ &= E_{\mathfrak{p}, \mathfrak{p}^-} \circ (\text{Ad } u \circ E_{\mathfrak{p}, \mathfrak{p}^-}) \circ \text{Ad } v \quad \text{since } \mathfrak{p}^u = \mathfrak{p}, \\ &= E_{\mathfrak{p}, \mathfrak{p}^-} \circ E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } v \quad \text{since } u \in U, \\ &= (E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } v) \circ E_{\mathfrak{p}^v, \mathfrak{p}^-} \\ &= E_{\mathfrak{p}, \mathfrak{p}^-} \circ E_{\mathfrak{p}^v, \mathfrak{p}^-} \in M \quad \text{since } v \in U^-. \end{aligned}$$

Hence $UU^- \subseteq \mathfrak{A}$. Similarly, since $E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } v = E_{\mathfrak{p}, \mathfrak{p}^-}$ for $v \in U^-$, it follows that $U^-UU^- \subseteq \mathfrak{A}$.

Since $\mathfrak{p}$ is maximal, by Proposition 2.5 we have $L \subseteq \mathfrak{A}_1$. But for $x \in L$, we have

$$E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } x \circ E_{\mathfrak{p}, \mathfrak{p}^-} = E_{\mathfrak{p}, \mathfrak{p}^-} \circ E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } x = E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } x.$$

Hence it follows that $L \subseteq \mathfrak{A}$. Now since $(L \cap \mathfrak{A})\mathfrak{A} \subseteq \mathfrak{A}$, we have $LU^-UU^- \subseteq \mathfrak{A}$. Hence $G \subseteq \mathfrak{A}$. So $\mathfrak{A} = G$. $\square$

Theorem 2.7. $M = \{1\} \cup \{E \circ \text{Ad } x \circ E' | E, E' \in \mathcal{E} \setminus \{1\}, x \in G\}$.

Proof. Let $N = \{1\} \cup \{E \circ \text{Ad } x \circ E' | E, E' \in \mathcal{E} \setminus \{1\}, x \in G\}$. By Proposition 2.6, we have $N \subseteq M$. By Corollary 2.3, we have

$$M = \{E_1 \circ E_2 \circ \cdots \circ E_k | E_1 \leftrightarrow E_2 \leftrightarrow \cdots \leftrightarrow E_k, E_i \in \mathcal{E}\}.$$

Let $E_i = E_{p_i, p_i^-}$. Consider $E_1 \circ E_2 \circ E_3$ with $E_1 \leftrightarrow E_2 \leftrightarrow E_3$. Then $p_1^- \leftrightarrow p_2$ and $p_2^- \leftrightarrow p_3$. Let $l_1 \subseteq p_1^- \cap p_2$, $l_2 = p_2 \cap p_2^-$, and $l_3 \subseteq p_2^- \cap p_3$ be the common Levi factors. Let P_2 and P_2^- denote the parabolic subgroups associated with p_2 and p_2^- , respectively. Let $L_2 = P_2 \cap P_2^-$, $P_2 = L_2 U_2$, and $P_2^- = L_2 U_2^-$, where $U_2 = R_u(P_2)$ and $U_2^- = R_u(P_2^-)$. Then there exists $u \in U_2$ and $v \in U_2^-$ such that $l_1 = {}^u l_2$ and $l_2 = {}^v l_3$. Let $b \in \mathcal{B}$ and $b_1 = E_{p_3, p_3^-}(b)$. Then

$$\begin{aligned} E_{p_1, p_1^-} \circ E_{p_2, p_2^-}(b_1) &= E_{p_1, p_1^-} \circ \text{proj}_{p_2} \circ \text{proj}_{p_2^-}(b_1) \\ &= E_{p_1, p_1^-} \circ \text{proj}_{p_2}(l_2 \cap {}^v b_1 + \tau_n(p_2^-)) \\ &= E_{p_1, p_1^-}(l_1 \cap {}^{uv} b_1 + \tau_n(p_2)) \\ &= (E_{p_1, p_1^-} \circ \text{Ad}(uv))(b_1). \end{aligned}$$

Hence

$$E_{p_1, p_1^-} \circ E_{p_2, p_2^-} \circ E_{p_3, p_3^-} = E_{p_1, p_1^-} \circ \text{Ad}(uv) \circ E_{p_3, p_3^-}.$$

Also note that $l_1 \subseteq p_1^- \cap {}^{uv} p_3$, hence $p_1^- \leftrightarrow {}^{uv} p_3$. Assume that $E_1 \leftrightarrow E_2 \leftrightarrow \dots \leftrightarrow E_{k-1}$ and $E_1 \circ E_2 \circ \dots \circ E_{k-1} = E_1 \circ \text{Ad } x \circ E_{k-1}$, with $p_1^- \leftrightarrow {}^x p_{k-1}$, hence $p_1^{-x} \leftrightarrow p_{k-1}$. Consider $E_1 \circ E_2 \circ \dots \circ E_k$ with $E_1 \leftrightarrow E_2 \leftrightarrow \dots \leftrightarrow E_k$. Then using Lemma 2.1,

$$\begin{aligned} E_1 \circ E_2 \circ \dots \circ E_{k-1} \circ E_k &= E_1 \circ \text{Ad } x \circ E_{k-1} \circ E_k \\ &= E_{p_1, p_1^-} \circ \text{Ad } x \circ E_{p_{k-1}, p_{k-1}^-} \circ E_{p_k, p_k^-} \\ &= \text{Ad } x \circ E_{p_1^x, p_1^{-x}} \circ E_{p_{k-1}, p_{k-1}^-} \circ E_{p_k, p_k^-} \\ &= \text{Ad } x \circ E_{p_1^x, p_1^{-x}} \circ \text{Ad } y \circ E_{p_k, p_k^-} \quad \text{since } E_{p_1^x, p_1^{-x}} \leftrightarrow E_{p_{k-1}, p_{k-1}^-} \leftrightarrow E_{p_k, p_k^-} \\ &= E_{p_1, p_1^-} \circ \text{Ad}(xy) \circ E_{p_k, p_k^-} \quad \text{with } p_1^- \leftrightarrow {}^{xy} p_k. \end{aligned}$$

Hence by induction $E_1 \circ E_2 \circ \dots \circ E_k = E_1 \circ \text{Ad } x \circ E_k$, $x \in G$, for all k . This proves that $M \subseteq N$. Hence $M = N$. $\square$

Fix $p \in \mathcal{P}$ with opposite p^- . Let $E = E_{p, p^-}$ and $\mathfrak{l} = p \cap p^-$. Let P and P^- denote the parabolic subgroups associated with p and p^- , respectively, and $L = P \cap P^-$. Let $\text{St}_G(\mathfrak{l})$ denote the stabilizer of $\mathfrak{l}$ in G and let $H = \{x \in \text{St}_G(\mathfrak{l}) \mid b = {}^x b \text{ for all } b \in \mathcal{B}(\mathfrak{l})\}$. Let $\tilde{L} = \text{St}_G(\mathfrak{l})/H$. Then we have the following

Theorem 2.8. *The maximal subgroup of M with identity element $E = E_{p, p^-}$ is $\{E \circ \text{Ad } x \circ E \mid x \in \text{St}_G(\mathfrak{l})\}$ which is isomorphic to $\tilde{L}$ as above.*

Proof. The maximal subgroup of M with identity element $E = E_{p, p^-}$ is the same as the group of units K say, of EME . Let $a \in K$. Then as in the proof of Theorem 2.7, we have $a = E_{q, q^-} \circ \text{Ad } y \circ E_{q_1, q_1^-}$ for some $y \in G$, where $q \subseteq p$, $q^- \subseteq p^-$, $q_1 \subseteq p$, $q_1^- \subseteq p^-$, and $q^- \leftrightarrow {}^y q_1$. Since $a \in K$ is invertible and E is the identity in K , there exists $b \in K$ such that $a \circ b = E$ and $b \circ a = E$. Hence, by Corollary 2.4,

$$E_{q, q^-} = E_{q, q^-} \circ E = E_{q, q^-} \circ a \circ b = a \circ b = E.$$

Similarly,

$$E_{q_1, q_1^-} = E \circ E_{q_1, q_1^-} = b \circ a \circ E_{q_1, q_1^-} = b \circ a = E.$$

Hence $a = E \circ \text{Ad } y \circ E$ for some $y \in G$, and $\mathfrak{p}^- \leftrightarrow {}^y\mathfrak{p}$. Let $\mathfrak{l}_1 = \mathfrak{p}^- \cap {}^y\mathfrak{p}$. Let $U = R_u(P)$ and $U^- = R_u(P^-)$. Then there exists $u \in U$ and $v \in U^-$ such that $\mathfrak{l}_1 = {}^v\mathfrak{l}$ and $\mathfrak{l}_1^y = {}^v\mathfrak{l}$, hence ${}^y\mathfrak{l} = \mathfrak{l}_1$. Therefore, ${}^x\mathfrak{l} = \mathfrak{l}$, where $x = v^{-1}yu$ so $y = vxu^{-1}$. Hence $x \in \text{St}_G(\mathfrak{l})$ and

$$\begin{aligned} a &= E \circ \text{Ad } y \circ E = (E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } v) \circ \text{Ad } x \circ (\text{Ad } u^{-1} \circ E_{\mathfrak{p}, \mathfrak{p}^-}) \\ &= E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } x \circ E_{\mathfrak{p}, \mathfrak{p}^-} = E \circ \text{Ad } x \circ E, \end{aligned}$$

since $v \in U^-$ and $u^{-1} \in U$. Observe that $\mathfrak{l} \subseteq \mathfrak{p} \cap \mathfrak{p}^{-x} \cap {}^x\mathfrak{p}^-$. Now define the map $\theta: \text{St}_G(\mathfrak{l}) \rightarrow K$, by $\theta(x) = E \circ \text{Ad } x \circ E$. By the above discussion θ is onto. Furthermore, for $x, y \in \text{St}_G(\mathfrak{l})$.

$$\begin{aligned} \theta(x) \circ \theta(y) &= E \circ \text{Ad } x \circ E \circ E \circ \text{Ad } y \circ E \\ &= E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } x \circ E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad } y \circ E_{\mathfrak{p}, \mathfrak{p}^-} \\ &= \text{Ad}(xy) \circ E_{\mathfrak{p}^{xy}, \mathfrak{p}^{-xy}} \circ E_{\mathfrak{p}^y, \mathfrak{p}^{-y}} \circ E_{\mathfrak{p}, \mathfrak{p}^-} \quad \text{by Lemma 2.1,} \\ &= \text{Ad}(xy) \circ E_{\mathfrak{p}^{xy}, \mathfrak{p}^{-xy}} \circ E_{\mathfrak{p}, \mathfrak{p}^-} \\ &\quad \text{since } \mathfrak{p}^{xy}, \mathfrak{p}^{-xy}, \mathfrak{p}^y, \mathfrak{p}^{-y}, \mathfrak{p}, \mathfrak{p}^- \text{ have common Levi factor } \mathfrak{l}, \\ &= E_{\mathfrak{p}, \mathfrak{p}^-} \circ \text{Ad}(xy) \circ E_{\mathfrak{p}, \mathfrak{p}^-} \quad \text{by Lemma 2.1} \\ &= \theta(xy). \end{aligned}$$

Hence θ is a surjective homomorphism. Clearly, $\text{Ker } \theta = H$. Hence $K \cong \text{St}_G(\mathfrak{l})/H = \tilde{L}$. $\square$

We note that $\tilde{L}$ is closely related to the Levi subgroup L of G . We therefore see that the monoid M gives rise to groups closely related to Kac-Moody groups.

3. EXAMPLES

When $\mathfrak{g}$ is finite dimensional, M is exactly the monoid generated by $\text{proj}_{\mathfrak{p}} (\mathfrak{p} \in \mathcal{P})$ and is isomorphic to the monoid studied in [5, Theorem 2] for the associated simple algebraic group. The situation is quite different when $\mathfrak{g}$ is infinite dimensional. For example, consider the affine Kac-Moody Lie algebras $\mathfrak{g} = A_1^{(1)}$ with Dynkin diagram

(3.1)

Then $\mathfrak{g}$ has four standard positive parabolics, but the set $\mathcal{P}$ of all positive parabolics is of course infinite. The monoid generated by $\text{proj}_{\mathfrak{p}} (\mathfrak{p} \in \mathcal{P})$ is not that interesting in that the subgroups of this monoid are trivial. We, therefore, are naturally led to considering our monoid M generated by $E_{\mathfrak{p}, \mathfrak{p}^-} = \text{proj}_{\mathfrak{p}} \circ \text{proj}_{\mathfrak{p}^-}$. The only nontrivial subgroups of M are those corresponding to parabolic subalgebras $\mathfrak{p}$ obtained by deleting a node of the diagram (3.1). The associated Levi factor $\mathfrak{l}$ is a finite-dimensional Lie algebra with the simple part being $\mathfrak{sl}_2(\mathbb{C})$. The subgroup of M with identity element $E_{\mathfrak{p}, \mathfrak{p}^-}$ is just the projective group $\text{PGL}(2, \mathbb{C})$.

To get finite-dimensional Kac-Moody groups, we consider the hyperbolic Kac-Moody Lie algebra $\mathfrak{g}_1$ with Dynkin diagram

(3.2)

Let $M_1 = M(\mathfrak{g}_1)$ denote the corresponding monoid. Let $\mathfrak{p}$ denote the parabolic subalgebra obtained by deleting the right-most node in (3.2). Then the maximal subgroup of M_1 with identity $E_{\mathfrak{p}, \mathfrak{p}^-}$ is 'almost' the Kac-Moody group corresponding to (3.1).

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27695-8205