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A SMOOTH HOLOMORPHICALLY CONVEX DISC IN $\mathbf{C}^2$ THAT IS NOT LOCALLY POLYNOMIALLY CONVEX

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ABSTRACT. We construct a smooth embedded disc in $\mathbf{C}^2$ that is totally real except at one point p, is holomorphically convex, but fails to be locally polynomially or even rationally convex at p.

INTRODUCTION

A compact set $K \subset \mathbf{C}^n$ is said to be *holomorphically convex* if K is the intersection of Stein open sets (domains of holomorphy) containing K. Equivalently, K has a basis of Stein neighborhoods in $\mathbf{C}^n$. The holomorphic hull $\widehat{K}_{\mathscr{H}}$ is the smallest holomorphically convex compact set containing K.

Recall that the *polynomially convex hull* $\widehat{K}$ of K is the set

$$\left\{z \in \mathbf{C}^n : |f(z)| \le \sup_K |f|,\ f \text{ holomorphic polynomial}\right\}.$$

The *rationally convex hull* $\widehat{K}_{\mathscr{R}}$ of K is the set of all points $z \in \mathbf{C}^n$ with the property that every holomorphic polynomial f on $\mathbf{C}^n$ that vanishes at z also vanishes somewhere on K.

For every compact set K we have

$$\widehat{K}_{\mathscr{H}} \subset \widehat{K}_{\mathscr{R}} \subset \widehat{K}.$$

It is well known that these hulls are in general different even when K is a rather simple set, e.g., a smoothly embedded disc in $\mathbf{C}^2$. Hörmander and Wermer [6] gave an example of a smooth embedded disc in $\mathbf{C}^2$ that is totally real and therefore holomorphically convex, but it bounds an analytic disc and thus is not polynomially or even rationally convex. Recently Duval [3] gave an example of a smooth embedded Lagrangian disc in $\mathbf{C}^2$ that is per force rationally convex according to the main result of [3], but it fails to be polynomially convex. A Lagrangian disc does not bound any complex varieties with reasonably nice boundaries, and the existence of the nontrivial hull is due in this case to a certain linking property of analytic discs in the polynomial hull.

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It seems that the known examples of smooth surfaces M in $\mathbf{C}^2$ that are holomorphically convex are at least *locally polynomially convex* at each point, i.e., sufficiently small neighborhoods of each point in M are polynomially convex. This is the case for all surfaces with nondegenerate complex tangents in the sense of Bishop [1]: at every elliptic complex tangent there is a nontrivial local envelope of holomorphy [1], while totally real points and the hyperbolic complex tangents are locally polynomially convex [5].

In this article we construct a smooth embedded holomorphically convex disc in $\mathbf{C}^2$ that fails to be locally polynomially or even rationally convex.

Choose any smooth function $g\colon [0, \infty) \to \mathbf{R}$ with a sequence of simple zeros $a_1 > a_2 > a_3 > \cdots > 0$ converging to 0 (and with no other zeros). For instance, $g(t) = \exp(-1/t)\sin(1/t)$ will do. Set

$$h(z) = \bar{z}g(|z|^2)\exp(i|z|^2),$$

and let M be its graph over the unit disc

$$M = \{(z, h(z)) \in \mathbf{C}^2 : |z| \le 1\}.$$

Theorem. *The smooth disc $M \subset \mathbf{C}^2$ defined above satisfies*

(a) *M is totally real outside the origin,*
(b) *M is holomorphically convex, and*
(c) *M has no rationally convex neighborhood of* 0.

A theorem of Hörmander and Wermer [6] and Preskenis [7] implies the following

Corollary. *Every continuous function on M can be approximated uniformly on M by functions holomorphic near M.*

However, because of (c), there is no single Stein neighborhood Ω of M such that every continuous function on M would be the uniform limit of functions holomorphic on Ω.

The complex tangent $0 \in M$ is highly degenerate; in fact, h vanishes to infinite order at 0. We do not know whether an example of this kind exists with a real-analytic function h.

Proof of the theorem. A simple calculation shows that the graph M of a function $h\colon \mathbf{C} \to \mathbf{C}$ is totally real at a point $(z, h(z))$ if and only if $h_{\bar{z}}(z) = \partial h/\partial \bar{z}(z) \neq 0$. With h as above we have

$$h_{\bar{z}}(z) = \exp(i|z|^2)((|z|^2 g' + g) + i|z|^2 g),$$

where $g' = dg/dt$. Since g only has simple zeros, $h_{\bar{z}}$ is nonzero outside the origin, so property (a) holds.

Since $h(\sqrt{a_j}\exp(i\theta)) = 0$, M bounds the analytic disc

$$D_j = \{(z, 0) : |z| \le \sqrt{a_j}\},$$

hence $D_j \subset \widehat{M}$ for all j. Since the discs D_j shrink to the origin as $j \to \infty$, M has no polynomially convex neighborhood of the origin. Moreover, as the boundary curve bD_j also bounds the disc $M_{\sqrt{a_j}} = M \cap \{|z| \le \sqrt{a_j}\}$, D_j is contained in the rational hull of $M_{\sqrt{a_j}}$. Namely, if $A \subset \mathbf{C}^2$ is a complex algebraic curve that avoids bD_j and intersects the interior of D_j, then the intersection index $A \cdot D_j$ is positive (two complex varieties always intersect positively) and A has the same intersection index with $M_{\sqrt{a_j}}$. This proves (c).

We now turn to the proof of (b). First we compare the sizes of $h_{\bar{z}}$ and h_z. We have

$$h_z = \partial h/\partial z = \bar{z}^2 \exp(i|z|^2)(g' + ig)$$

and

$$|h_{\bar{z}}|^2 - |h_z|^2 = g^2 + 2|z|^2 gg'.$$

We can find points $b > 0$ arbitrarily close to 0 such that

(a) $g(\sqrt{b})g'(\sqrt{b}) > 0$ and
(b) $|g(t)| < |g(\sqrt{b})|$ for $0 \le t < \sqrt{b}$.

Fix a b_0 satisfying these properties and choose a $b_1 > b_0$ such that (a) and (b) hold for every $b \in [b_0, b_1]$. Notice that $|h(z)| = |z|\,|g(|z|^2)|$ is a radial function depending only on $|z|$. It follows that there is a constant $C > 0$ such that for all points z in the annulus $A(b_0, b_1) = \{b_0 \le |z| \le b_1\}$ we have

(i) $|h_{\bar{z}}|^2 - |h_z|^2 \ge C > 0$ and
(ii) $|h(z)|$ is a strictly increasing function of $|z|$.

Let P_j $(j = 0, 1)$ be the polydisc

$$P_j = \{(z, w): |z| \le b_j, \ |w| \le |h(b_j)|\}.$$

Set $K_0 = P_0$, $K_1 = (K_0 \cup M) \cap P_1 = K_0 \cup (M \cap P_1)$, and $S = K_0 \cup M$. Then $\widehat{K}_0 = K_0$, $S \backslash K_0$ is a totally real submanifold of $\mathbf{C}^2 \backslash K_0$, and K_1 is a relative neighborhood of K_0 in S.

Proposition. *The set K_1 is holomorphically convex (in fact, even polynomially convex).*

If the proposition holds, then a theorem of Hörmander and Wermer [6] implies that the set $S = K_0 \cup M$ is holomorphically convex, so the holomorphic hull of M is contained in $K_0 \cup M$. As $b > 0$ can be chosen arbitrarily small, the polydisc K_0 is arbitrarily small, hence M is holomorphically convex as claimed. This proves our theorem, provided that the proposition holds.

Proof of the proposition. The proof is inspired by Duval [2, 3] and Preskenis [7]. Let

$$\Delta_+(\varepsilon) = \{\zeta \in \mathbf{C}: |\zeta| \le \varepsilon, \ \Re\zeta > 0\}.$$

For each $a \in \mathbf{C}$, $|a| \le 1$, we set

$$Q_a(z, w) = (z - a)(w - h(a)).$$

In order to complete this proof, we need the following

Lemma. *For each $b_2 > 0$ satisfying $b_0 < b_2 < b_1$ there is an $\varepsilon_0 > 0$ such that for every $a \in A(b_2, b_1)$ and for every $\alpha \in \Delta_+(\varepsilon_0)$ the quadric $\mathcal{V}_{a,\alpha} \subset \mathbf{C}^2$, defined by the equation*

$$Q_a(z, w) + \alpha h_{\bar{z}}(a) = 0,$$

avoids K_1.

Proof of the lemma. Using the Taylor expansion of $h(z)$ at a we get

$$\begin{aligned} &Q_a(z, h(z)) + \alpha h_{\bar{z}}(a) \\ &\quad = (z - a)\,(h_{\bar{z}}(a)(\bar{z} - \bar{a}) + h_z(a)(z - a)) + \alpha h_{\bar{z}}(a) + o(|z - a|^2) \\ &\quad = h_{\bar{z}}(a)(|z - a|^2 + \alpha) + (z - a)^2 h_z(a) + o(|z - a|^2). \end{aligned}$$

Since $|h_{\bar z}(a)| > |h_z(a)|$, this expression is nonvanishing near $z = a$ for every α with $\Re\alpha > 0$. Thus there are a neighborhood V of $(a, h(a))$ with size depending only on a (and of course on h) and an $\varepsilon_0 > 0$ such that for every $\alpha \in \Delta_+(\varepsilon_0)$ we have $\mathscr{V}_{a,\alpha} \cap K_1 \cap V = \varnothing$.

As α tends to zero, the quadric $\mathscr{V}_{a,\alpha}$ tends to $Q_a(z, w) = 0$, uniformly outside V. Since the quadric $Q_a(z, w) = 0$ intersects K_1 only at the point $(a, h(a))$, we can decrease ε_0 if necessary to ensure that $\mathscr{V}_{a,\alpha} \cap K_1 = \varnothing$ whenever $\alpha \in \Delta_+(\varepsilon_0)$. The construction shows that we can choose $\varepsilon_0 > 0$ independent of $a \in A(b_2, b_1)$. This proves the lemma.

Fix a point $(z_0, w_0) \in P_1 \backslash K_1$. We shall find a quadric $\mathscr{V}_{a,\alpha}$ passing through (z_0, w_0) and avoiding K_1. This will imply that K_1 is rationally convex and therefore holomorphically convex. An additional argument as in [2] shows that K_1 is polynomially convex, but we shall not need this fact.

At least one of the lines $z = z_0$, $w = w_0$ avoids the polydisc P_0. Suppose that $z = z_0$ does, as the proof in the other case is completely analogous. The property (b) (§2) and the definition of h show that there is a unique point $z_1 \in A(b_0, b_1)$ satisfying $h(z_1) = w_0$. Choose b_2 such that $b_0 < b_2 < |z_1| \le b_1$, and choose an $\varepsilon_0 > 0$ such that the lemma holds on $A(b_2, b_1)$. To conclude the proof it suffices to find an a close to z_1, with $b_2 \le |a| \le b_1$, and an $\alpha \in \Delta_+(\varepsilon_0)$ such that $\mathscr{V}_{a,\alpha}$ passes through (z_0, w_0). (Recall that this quadric avoids K_1 by construction.)

The last condition means

$$(z_0 - a)(w_0 - h(a)) + \alpha h_{\bar z}(a) = 0.$$

This is satisfied if we set

$$\alpha = (z_0 - a)(h(a) - w_0)/h_{\bar z}(a).$$

It remains to choose $a = z_1 + \zeta$, with ζ sufficiently small, such that $\alpha \in \Delta_+(\varepsilon_0)$.

Using the Taylor expansion for $h(a)$ at the point z_1 we get

$$\begin{aligned}\alpha &= (z_0 - z_1)(h_{\bar z}(z_1)\bar\zeta + h_z(z_1)\zeta)/h_{\bar z}(z_1) + o(|\zeta|)\\ &= \bar\zeta(z_0 - z_1)(1 + \zeta h_z(z_1)/\bar\zeta h_{\bar z}(z_1)) + o(|\zeta|).\end{aligned}$$

Since $|h_z/h_{\bar z}| < 1$, we get for $\zeta = \varepsilon/(\bar z_0 - \bar z_1)$, with $\varepsilon > 0$ sufficiently small, that $\alpha \in \Delta_+(\varepsilon_0)$ and $a = z_1 + \zeta \in A(b_2, b_1)$. This concludes the proof of the proposition.

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