LINEAR TRANSFORMATIONS OF WIENER INTEGRALS

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ABSTRACT. In this paper we obtain a linear transformation theorem in which the Radon-Nikodym derivative is very closely related to the transformation. We also obtain a vector-valued conditional version of this linear transformation theorem.

1. INTRODUCTION

In [1, 2] Cameron and Martin investigated various linear transformations of the standard Wiener measure. Since then many related papers, including [3, 6, 7, 11–17], have appeared in the literature. In particular, [12] has a discussion of previous work and many additional references.

Let $C[0, T]$ be the space of all real-valued continuous functions $x(t)$ on $[0, T]$ with $x(0) = 0$. Let μ be a Gaussian measure defined on $C[0, T]$. Such a measure μ is uniquely determined by its mean m and covariance R , where

$$m(t) = \int_{C[0, T]} x(t) \mu(dx)$$

and

$$R(s, t) = \int_{C[0, T]} [x(s) - m(s)][x(t) - m(t)] \mu(dx).$$

The standard Wiener measure μ_w is the Gaussian measure on $C[0, T]$ with mean 0 and covariance function $R(s, t) = \min(s, t)$.

Suppose μ is a Gaussian measure on $C[0, T]$ with mean $m(t)$ and covariance $R(s, t)$. According to Shepp [12], μ and μ_w are equivalent, $\mu \sim \mu_w$, if and only if there exists a symmetric function $K(s, t) \in L^2([0, T]^2)$ for which

$$(1.1) \quad R(s, t) = \min(s, t) - \int_0^s \int_0^t K(u, v) du dv,$$

$$(1.2) \quad 1 \notin \sigma(K) \equiv \left\{ \lambda: \int_0^T K(t, u) \phi(u) du = \lambda \phi(t), \phi \in L^2[0, t], \|\phi\| = 1 \right\},$$

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and

$$(1.3) \quad m(t) = \int_0^t k(u) du \quad \text{for some } k \in L^2[0, T].$$

Let $\phi_1, \phi_2, \dots$ be the orthonormal eigenfunctions of K with eigenvalues $\lambda_1, \lambda_2, \dots$. Then $\{\phi_1, \phi_2, \dots\}$ is a complete orthonormal set in $L^2[0, T]$ with $\lambda_j < 1$ for $j = 1, 2, \dots$ and K has the L^2 -expansion

$$(1.4) \quad K(s, t) = \sum_{j=1}^{\infty} \lambda_j \phi_j(s) \phi_j(t).$$

If $\sum_{j=1}^{\infty} |\lambda_j| < \infty$, then K is said to be of trace class and the Fredholm determinant $d(1)$ is given by

$$(1.5) \quad d(1) = \prod_{j=1}^{\infty} (1 - \lambda_j).$$

Define $H(s, t)$ and $M(s, t)$ respectively, by

$$(1.6) \quad H(s, t) = \sum_{j=1}^{\infty} \frac{\lambda_j}{1 - \lambda_j} \phi_j(s) \phi_j(t)$$

and

$$(1.7) \quad M(s, t) = \sum_{j=1}^{\infty} (1 - \sqrt{1 - \lambda_j}) \phi_j(s) \phi_j(t).$$

Using Shepp's Radon-Nikodym derivatives of Gaussian measures [12], we state the following:

Theorem A. Suppose K is of trace class. Let S be a Wiener measurable subset of $C[0, T]$, and let $T_1 S$ be the image of S under the linear transformation

$$(1.8) \quad T_1: y(t) = x(t) - \int_0^t \int_0^T M(s, u) dx(u) ds.$$

If F is any Wiener measurable function, then

$$\begin{aligned} & \int_{T_1 S} F(y) \exp \left\{ -\frac{1}{2} \int_0^T \int_0^T H(s, t) dy(s) dy(t) \right\} \mu_w(dy) \\ &= [d(1)]^{1/2} \int_S F \left(x(\cdot) - \int_0^\cdot \int_0^T M(s, u) dx(u) ds \right) \mu_w(dx), \end{aligned}$$

in the sense the existence of either side implies that of the other side and their equality.

The corresponding, but less general, result by Cameron and Martin [2] states essentially

Theorem B. Let $U(t, s)$ be an appropriate L^2 -function on $[0, T]^2$. Let S be a Wiener measurable subset of $C[0, T]$, and let $T_2 S$ be the image of S under the linear transformation

$$(1.9) \quad T_2: y(t) = x(t) + \int_0^T U(t, u) x(u) du.$$

If $F(y)$ is any Wiener measurable function for which either side of (1.10) exists, then the other side also exists and the equality holds,

$$(1.10) \quad \int_{T_2 S} F(y) \mu_w(dy) = |D| \int_S F \left(x(\cdot) + \int_0^T U(\cdot, u) x(u) du \right) \times \exp \left\{ -\frac{1}{2} \Phi(x) \right\} \mu_w(dx),$$

where

$$(1.11) \quad D = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^T \cdots \int_0^T \left| \frac{U(s_1, s_1) \cdots U(s_1, s_n)}{U(s_n, s_1) \cdots U(s_n, s_n)} \right| ds_1 \cdots ds_n$$

and

$$(1.12) \quad \begin{aligned} \Phi(x) = & \int_0^T \left[\frac{d}{dt} \int_0^T U(t, u) x(u) du \right]^2 dt \\ & + 2 \int_0^T \left[\frac{d}{dt} \int_0^T U(t, u) x(u) du \right] dx(t). \end{aligned}$$

Note that Theorem B is a modern version of Cameron and Martin's theorem in [2] as the concept of stochastic integrals was not fully developed at that time. Also note that the transformation (1.9) is less general than the linear transformation (1.8). Indeed, if we set $U(t, u) = \int_0^t V(s, u) ds$, then

$$x(t) + \int_0^T U(t, u) x(u) du = x(t) - \int_0^t \int_0^T \left[\int_T^u V(s, v) dv \right] dx(u) ds.$$

However, Theorem B is easier to apply because the Radon-Nikodym derivative depends on $U(t, s)$, while in Theorem A, the Radon-Nikodym derivative depends on another function $H(s, t)$ that is often not easy to find in a closed form.

One main purpose of this paper is to find another version of Theorem A in which the Radon-Nikodym derivative is more closely related to the transformation. In doing so, we are able to show that Theorems A and B are more closely related than they appear to be; see Theorems 1 and 2 below.

In §3 below we obtain a linear transformation theorem for vector-valued conditional Wiener integrals.

2. MAIN RESULTS

The following theorem, which is a substantial improvement over Theorem A, plays a key role in this paper.

Theorem 1. Let $N(s, t) \in L^2([0, T]^2)$. Let $\{x(t): 0 \leq t \leq T\}$ be the standard Wiener process, and let $y(t)$ be defined on $[0, T]$ by

$$(2.1) \quad y(t) = x(t) - \int_0^t \int_0^T N(s, u) dx(u) ds.$$

Let $R(s, t)$ with kernel $K(s, t) = (-\partial^2 / \partial s \partial t) R(s, t)$ a.e. on $[0, T]^2$ denote the covariance of $y(t)$. If $K(s, t)$ is of trace class with $1 \notin \sigma(K)$, then for any

Wiener measurable set S in $C[0, T]$ and any Wiener measurable function F , we have

$$(2.2) \quad \int_{T_3 S} F(y) \mu_w(dy) = [d(1)]^{1/2} \int_S F \left(x(\cdot) - \int_0^\cdot \int_0^T N(s, u) dx(u) ds \right) \\ \times \exp \left\{ \frac{1}{2} \int_0^T \int_0^T K(s, t) dx(s) dx(t) \right\} \mu_w(dx),$$

in the sense the existence of either side implies that of the other side and their equality, where T_3 is the linear transformation given by (2.1).

Proof. First write

$$(2.3) \quad \int_{T_3 S} F(y) \mu_w(dy) = \int_{T_3 S} F(y) \exp \left\{ \frac{1}{2} \int_0^T \int_0^T H(s, t) dy(s) dy(t) \right\} \\ \times \exp \left\{ -\frac{1}{2} \int_0^T \int_0^T H(s, t) dy(s) dy(t) \right\} \mu_w(dy).$$

We next apply Theorem A to the right-hand side of (2.3) and obtain

$$(2.4) \quad \int_{T_3 S} F(y) \mu_w(dy) = [d(1)]^{1/2} \int_S F \left(x(\cdot) - \int_0^\cdot \int_0^T M(s, u) dx(u) ds \right) \\ \times \exp \left\{ \frac{1}{2} \int_0^T \int_0^T H(s, t) d \left[x(s) - \int_0^s \int_0^T M(v, u) dx(u) dv \right] \right. \\ \left. \times d \left[x(t) - \int_0^t \int_0^T M(v, u) dx(u) dv \right] \right\} \mu_w(dx).$$

Using the expression (1.6), (1.7), and (1.4) for H , M , and K , a straightforward computation yields that

$$(2.5) \quad \int_0^T \int_0^T H(s, t) d \left[x(s) - \int_0^s \int_0^T M(v, u) dx(u) dv \right] \\ \times d \left[x(t) - \int_0^t \int_0^T M(v, u) dx(u) dv \right] \\ = \int_0^T \int_0^T K(s, t) dx(s) dx(t).$$

Thus, we have

$$(2.6) \quad \int_{T_3 S} F(y) \mu_w(dy) = [d(1)]^{1/2} \int_S F \left(x(\cdot) - \int_0^\cdot \int_0^T M(s, u) dx(u) ds \right) \\ \times \exp \left\{ \frac{1}{2} \int_0^T \int_0^T K(s, t) dx(s) dx(t) \right\} \mu_w(dx).$$

But $x(t) - \int_0^t \int_0^T M(s, u) dx(u) ds$ and $x(t) - \int_0^t \int_0^T N(s, u) dx(u) ds$ are equivalent processes since they have the same covariance function, namely, R , and

the same mean function, namely, the zero function. Thus, they have the same Radon-Nikodym derivative, and so (2.2) follows from (2.6).

Remark. Note that in Theorem 1 we did not need to require $N(s, t)$ to be a symmetric function. However, $M(s, t)$ is symmetric as is easily seen by (1.7).

In our next theorem we show that if $N(s, t)$ also satisfies the condition

$$(2.7) \quad \int_0^T N(s, u)N(t, u) du = \int_0^T N(u, s)N(u, t) du$$

then the Radon-Nikodym derivative has a Cameron-Martin type expression. Thus Theorems A and B are closely related for a large class of linear transformations.

Theorem 2. Let $N(s, t)$, $K(s, t)$, S , and F satisfy the hypotheses in Theorem 1. Furthermore, assume that $N(s, t)$ satisfies (2.7) a.e. on $[0, T]^2$. Let T_3 be the linear transformation given by (2.1). Then we have that

$$\begin{aligned} \int_{T_3 S} F(y) \mu_w(dy) &= [d(1)]^{1/2} \int_S F \left(x(\cdot) - \int_t^{\cdot} \int_0^T N(s, u) dx(u) ds \right) \\ &\quad \times \exp \left\{ -\frac{1}{2} \Psi(x) \right\} \mu_w(dx), \end{aligned}$$

where

$$\begin{aligned} \Psi(x) &= \int_0^T \left[\frac{d}{dt} \left(- \int_0^t \int_0^T N(s, u) dx(u) ds \right) \right]^2 dt \\ &\quad + 2 \int_0^T \left[\frac{d}{dt} \left(- \int_0^t \int_0^T N(s, u) dx(u) ds \right) \right] dx(t). \end{aligned}$$

Proof. By comparing Theorem 2 with Theorem 1 we see that it is sufficient to show that

$$\Psi(x) = - \int_0^T \int_0^T K(s, t) dx(s) dx(t).$$

But it is easy to see that

$$\begin{aligned} \Psi(x) &= \int_0^T \left[\int_0^T N(t, u) dx(u) \right]^2 dt - 2 \int_0^T \int_0^T N(t, u) dx(u) dx(t) \\ (2.8) \quad &= \int_0^T \int_0^T \left[\int_0^T N(t, u) N(t, v) dt \right] dx(u) dx(v) \\ &\quad - 2 \int_0^T \int_0^T N(v, u) dx(u) dx(v). \end{aligned}$$

But, using the Fubini Theorem, and then simply interchanging u and v , we see that

$$\begin{aligned} \int_0^T \int_0^T N(v, u) dx(u) dx(v) &= \int_0^T \int_0^T N(v, u) dx(v) dx(u) \\ (2.9) \quad &= \int_0^T \int_0^T N(u, v) dx(u) dx(v). \end{aligned}$$

Thus, using (2.7) and (2.9) in (2.8), we obtain

$$\Psi(x) = \int_0^T \int_0^T \left[\int_0^T N(u, t)N(v, t) dt - N(v, u) - N(u, v) \right] dx(u) dx(v).$$

On the other hand, see Shepp [12, p. 331], $K(u, v)$ and $N(u, v)$ are related by

$$(2.10) \quad K(u, v) = N(u, v) + N(v, u) - \int_0^T N(u, t)N(v, t) dt.$$

Hence, $\Psi(x) = -\int_0^T \int_0^T K(u, v) dx(u) dx(v)$, and the proof is complete.

Next we give two examples. In Example 1, $N(s, t)$ satisfies the conditions of Theorem 1 but not those of Theorem 2, i.e., not condition (2.7). In Example 2, $N(s, t)$ is not symmetric but it satisfies the conditions of Theorem 2.

Example 1. For convenience we choose $T = 1$. Let $\phi_1(s) = \sqrt{2}X_{[0, 1/2]}(s)$, let $\phi_2(s) = \sqrt{2}X_{(1/2, 1]}(s)$, and let

$$N(s, t) = \frac{1}{2}\phi_1(s)\phi_1(t) + \frac{2}{3}\phi_2(s)\phi_2(t) + \frac{3}{4}\phi_1(s)\phi_2(t) - \frac{1}{2}\phi_2(s)\phi_1(t).$$

$N(s, t)$ satisfies the conditions of Theorem 1 since by the use of (2.10),

$$K(s, t) = \frac{3}{16}\phi_1(s)\phi_1(t) + \frac{23}{36}\phi_2(s)\phi_2(t),$$

and so

$$d(1) = \left(1 - \frac{3}{16}\right) \left(1 - \frac{23}{36}\right) = \frac{169}{576}.$$

Also

$$M(s, t) = \frac{4 - \sqrt{13}}{4}\phi_1(s)\phi_1(t) + \frac{6 - \sqrt{13}}{6}\phi_2(s)\phi_2(t).$$

However

$$\begin{aligned} & \int_0^1 [N(s, u)N(t, u) - N(u, s)N(u, t)] du \\ &= \begin{cases} \frac{5}{8}, & (s, t) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right], \\ -\frac{5}{8}, & (s, t) \in \left(\frac{1}{2}, 1\right] \times \left(\frac{1}{2}, 1\right], \\ \frac{5}{12}, & \text{elsewhere in } [0, 1] \times [0, 1], \end{cases} \end{aligned}$$

and so N does not satisfy condition (2.7).

Example 2. Let $\phi_1(s)$ and $\phi_2(s)$ be any orthonormal functions on $[0, T]$, and let

$$N(s, t) = \phi_1(s)\phi_1(t) + \phi_2(s)\phi_2(t) + \frac{\sqrt{3}}{2}(\phi_1(s)\phi_2(t) - \phi_2(s)\phi_1(t)).$$

Clearly $N(s, t)$ is not symmetric. However, N satisfies the conditions of Theorem 2 since

$$\begin{aligned} K(s, t) &= \frac{1}{4}\phi_1(s)\phi_1(t) + \frac{1}{4}\phi_2(s)\phi_2(t), \\ d(1) &= \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{4}\right) = \frac{9}{16}, \\ M(s, t) &= \frac{2 - \sqrt{3}}{2}(\phi_1(s)\phi_1(t) + \phi_2(s)\phi_2(t)), \end{aligned}$$

and

$$\int_0^T N(s, u)N(t, u) du = \int_0^T N(u, s)N(u, t) du.$$

The following are natural generalizations of Theorems 1 and 2.

Theorem 3. Let $N(s, t)$, $K(s, t)$, S , and F be as in Theorem 1, and let $m(t)$ satisfy (1.3). Then, under the transformation

$$(2.11) \quad L: y(t) = x(t) - \int_0^t \int_0^T N(s, u) dx(u) ds + m(t),$$

we have

$$\begin{aligned} (2.12) \quad & \int_{LS} F(y) \mu_w(dy) \\ &= [d(1)]^{1/2} \int_S F\left(x(\cdot) - \int_0^\cdot \int_0^T N(s, u) dx(u) ds + m(\cdot)\right) \\ & \quad \times \exp\left\{-\frac{1}{2}\Phi(x)\right\} \mu_w(dx) \end{aligned}$$

where

$$\begin{aligned} \Phi(x) &= \int_0^T k^2(t) dt + 2 \int_0^T k(t) dx(t) - 2 \int_0^T k(t) \int_0^T N(t, u) dx(u) dt \\ & \quad - \int_0^T \int_0^T K(s, t) dx(s) dx(t). \end{aligned}$$

Proof. Let $L_1: z(t) = x(t) - \int_0^t \int_0^T N(s, u) dx(u) ds$ and $L'_1: y(t) = z(t) + m(t)$. Then $LS = L'_1 L_1 S$. Using the Cameron-Martin translation theorem [1], we obtain

$$\begin{aligned} (2.13) \quad & \int_{LS} F(y) \mu_w(dy) = \int_{L'_1 L_1 S} F(y) \mu_w(dy) \\ &= \exp\left\{-\frac{1}{2} \int_0^T k^2(t) dt\right\} \int_{L_1 S} F(z(\cdot) + m(\cdot)) \\ & \quad \times \exp\left\{-\int_0^T k(t) dz(t)\right\} \mu_w(dz). \end{aligned}$$

Next using Theorem 1, we have

$$\begin{aligned}
 & \int_{L_1 S} F(z(\cdot) + m(\cdot)) \exp \left\{ - \int_0^T k(t) dz(t) \right\} \mu_w(dz) \\
 &= [d(1)]^{1/2} \int_S F \left(x(\cdot) - \int_0^\cdot \int_0^T N(s, u) dx(u) ds + m(\cdot) \right) \\
 (2.14) \quad & \times \exp \left\{ - \int_0^T k(t) d \left[x(t) - \int_0^t \int_0^T N(s, u) dx(u) ds \right] \right\} \\
 & \times \exp \left\{ \frac{1}{2} \int_0^T \int_0^T K(s, t) dx(s) dx(t) \right\} \mu_w(dx).
 \end{aligned}$$

Simplification of the right-hand side of (2.14) together with (2.13) establishes (2.12).

The proof of the following theorem will not be included since it is similar to that of Theorem 2, but where we use Theorem 3 instead of Theorem 1.

Theorem 4. Let $N(s, t)$, $K(s, t)$, S , and F be as in Theorem 2, and let $m(t)$ satisfy (1.3). Then, under the transformation

$$L: y(t) = x(t) - \int_0^t \int_0^T N(s, u) dx(u) ds + m(t),$$

we have

$$\begin{aligned}
 \int_{LS} F(y) \mu_w(dy) &= [d(1)]^{1/2} \int_S F \left(x(\cdot) - \int_0^\cdot \int_0^T N(s, u) dx(u) ds + m(\cdot) \right) \\
 & \times \exp \left\{ - \frac{1}{2} \Psi_1(x) \right\} \mu_w(dx)
 \end{aligned}$$

where

$$\begin{aligned}
 \Psi_1(x) &= \int_0^T \left[\frac{d}{dt} \left(m(t) - \int_0^t \int_0^T N(s, u) dx(u) ds \right) \right]^2 dt \\
 &+ 2 \int_0^T \left[\frac{d}{dt} \left(m(t) - \int_0^t \int_0^T N(s, u) dx(u) ds \right) \right] dx(t).
 \end{aligned}$$

3. CONDITIONAL FORM OF THE LINEAR TRANSFORMATIONS

Let $\{x(t): 0 \leq t \leq T\}$ be the standard Wiener process, and let $\{y(t): 0 \leq t \leq T\}$ be a Gaussian process with mean zero and covariance $R(s, t)$ with kernel function $K(s, t) = (-\partial^2/\partial s \partial t)R(s, t)$. Suppose $1 \notin \sigma(K)$. Let $\{\phi_n\}$ be the orthonormal eigenfunctions of $K(s, t)$ so that

$$(3.1) \quad K(s, t) = \sum_{j=1}^{\infty} \lambda_j \phi_j(s) \phi_j(t), \quad \lambda_j < 1 \text{ for all } j.$$

Then $x(t)$ and $y(t)$ can be expressed in the form

(3.2)

$$x(t) = \sum_{j=1}^{\infty} \Phi_j(t) \int_0^T \phi_j(s) dx(s), \quad \Phi_j(t) = \int_0^t \phi_j(s) ds \quad \text{for } 0 \leq t \leq T,$$

(3.3)

$$y(t) = \sum_{j=1}^{\infty} \sqrt{1 - \lambda_j} \Phi_j(t) \int_0^T \phi_j(s) dx(s).$$

Assume that $K(s, t)$ is of trace class, and set

(3.4)

$$M(s, t) = \sum_{j=1}^{\infty} (1 - \sqrt{1 - \lambda_j}) \phi_j(s) \phi_j(t).$$

Then $y(t)$ has the representation

(3.5)

$$y(t) = x(t) - \int_0^t \int_0^T M(s, u) dx(u) ds,$$

from which it follows that

(3.6)

$$\int_0^T \phi_j(s) dy(s) = \sqrt{1 - \lambda_j} \int_0^T \phi_j(s) dx(s), \quad j = 1, 2, \dots$$

The following result is an immediate consequence of the above statements.

Proposition 1. *If $M(s, t)$ satisfies (3.4), then*

$$x(t) - \int_0^t \int_0^T M(s, u) dx(u) ds - \sum_{j=1}^n \Phi_j(t) \sqrt{1 - \lambda_j} \int_0^T \phi_j(u) dx(u)$$

and $\int_0^T \phi_k(u) dx(u)$ are stochastically independent for $k = 1, 2, \dots, n$.

The following theorem is the conditional form of Theorem 1.

Theorem 5. *Let x, y, K , and M satisfy conditions (3.1) through (3.5). Then, for any Wiener integrable function F , the conditional Wiener integrals satisfy*

$$\begin{aligned} E \left[F(y) \middle| \int_0^T \phi_j(u) dy(u) = \xi_j, \quad j = 1, \dots, n \right] \\ = \left(\prod_{r=n+1}^{\infty} (1 - \lambda_r)^{1/2} \right) \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{\lambda_j}{1 - \lambda_j} \xi_j^2 \right\} \\ \times E \left[F \left(x(\cdot) - \int_0^{\cdot} \int_0^T M(s, u) dx(u) ds \right) \right. \\ \left. \times \exp \left\{ \frac{1}{2} \int_0^T \int_0^T K(s, t) dx(s) dx(t) \right\} \middle| \int_0^T \phi_j(u) dx(u) \right. \\ \left. = \frac{\xi_j}{\sqrt{1 - \lambda_j}}, \quad j = 1, \dots, n \right]. \end{aligned}$$

Proof. In [10] we showed that

$$(3.7) \quad \begin{aligned} E \left[F(y) \middle| \int_0^T \phi_j(u) dy(u) = \xi_j, \quad j = 1, \dots, n \right] \\ = E \left[F(y(\cdot) - \sum_{j=1}^n \Phi_j(\cdot) \int_0^T \phi_j dy + \sum_{j=1}^n \Phi_j(\cdot) \xi_j \right] \end{aligned}$$

where

$$\Phi_j(t) = \int_0^t \phi_j(u) du \quad \text{and} \quad \int_0^T \phi_j dy = \int_0^T \phi_j(u) dy(u)$$

for $j = 1, \dots, n$. Next, an application of Theorem 1 (with $N(s, t)$ replaced by $M(s, t)$) on the right-hand side of (3.7) together with (3.6) yields

$$(3.8) \quad \begin{aligned} E \left[F(y) \middle| \int_0^T \phi_j dy = \xi_j, \quad j = 1, \dots, n \right] \\ = [d(1)]^{1/2} E \left[F \left(x(\cdot) - \int_0^\cdot \int_0^T M(s, u) dx(u) ds \right. \right. \\ \left. \left. - \sum_{j=1}^n \Phi_j(\cdot) \sqrt{1 - \lambda_j} \int_0^T \phi_j dx + \sum_{j=1}^\infty \Phi_j(\cdot) \xi_j \right) \right. \\ \left. \times \exp \left\{ \frac{1}{2} \int_0^T \int_0^T K(s, t) dx(s) dx(t) \right\} \right]. \end{aligned}$$

Next, using (3.1) and the fact that the sequence $\{\phi_n\}$ is orthonormal, a straightforward but lengthy calculation shows that

$$(3.9) \quad \begin{aligned} \int_0^T \int_0^T K(s, t) dx(s) dx(t) + \sum_{j=1}^n \frac{\lambda_j}{1 - \lambda_j} \xi_j^2 - \sum_{j=1}^n \lambda_j \left(\int_0^T \phi_j dx \right)^2 \\ = \int_0^T \int_0^T K(s, t) d \left[x(s) - \sum_{j=1}^n \Phi_j(s) \int_0^T \phi_j dx + \sum_{j=1}^n \Phi_j(s) \frac{\xi_j}{\sqrt{1 - \lambda_j}} \right] \\ \times d \left[x(t) - \sum_{j=1}^n \Phi_j(t) \int_0^T \phi_j dx + \sum_{j=1}^n \Phi_j(t) \frac{\xi_j}{\sqrt{1 - \lambda_j}} \right]. \end{aligned}$$

In view of the fact that $\{\int_0^T \phi_j dx, \quad j = 1, \dots, n\}$ is independent of $x(t) - \int_0^t \int_0^T M(s, u) dx(u) ds - \sum_{j=1}^n \Phi_j(t) \sqrt{1 - \lambda_j} \int_0^T \phi_j dx$ and also of $x(t) - \sum_{j=1}^n \Phi_j(t) \int_0^T \phi_j dx$, the first by Proposition 1, we may, using (3.9) rewrite

the right-hand side of (3.8) in the form

$$\begin{aligned}
 & [d(1)]^{1/2} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{\lambda_j}{1-\lambda_j} \xi_j^2 \right\} E \left[\exp \left\{ \frac{1}{2} \sum_{j=1}^n \lambda_j \left(\int_0^T \phi_j dx \right)^2 \right\} \right] \\
 & \times E \left[F \left(x(\cdot) - \int_0^\cdot \int_0^T M(s, u) dx(u) ds \right. \right. \\
 & \quad \left. \left. - \sum_{j=1}^n \Phi_j(\cdot) \sqrt{1-\lambda_j} \int_0^T \phi_j dx + \sum_{j=1}^n \Phi_j(\cdot) \xi_j \right) \right. \\
 (3.10) \quad & \times \exp \left\{ \frac{1}{2} \int_0^T \int_0^T K(s, t) d \left[x(s) - \sum_{j=1}^n \Phi_j(s) \int_0^T \phi_j dx \right. \right. \\
 & \quad \left. \left. + \sum_{j=1}^n \Phi_j(s) \frac{\xi_j}{\sqrt{1-\lambda_j}} \right] \right. \\
 & \times \left. \left. d \left[x(t) - \sum_{j=1}^n \Phi_j(t) \int_0^T \phi_j dx + \sum_{j=1}^n \Phi_j(t) \frac{\xi_j}{\sqrt{1-\lambda_j}} \right] \right\} \right]
 \end{aligned}$$

The last Wiener integral (expected value) is clearly equal to the conditional Wiener integral

$$\begin{aligned}
 & E \left[F \left(x(\cdot) - \int_0^\cdot \int_0^T M(s, u) dx(u) ds \right) \right. \\
 (3.11) \quad & \times \exp \left\{ \frac{1}{2} \int_0^T \int_0^T K(s, t) dx(s) dx(t) \right\} \left| \int_0^T \phi_j dx \right. \\
 & \quad \left. = \frac{\xi_j}{\sqrt{1-\lambda_j}}, j = 1, \dots, n \right],
 \end{aligned}$$

and by a well-known Wiener integration formula,

$$(3.12) \quad E \left[\exp \left\{ \frac{1}{2} \sum_{j=1}^n \lambda_j \left(\int_0^T \phi_j dx \right)^2 \right\} \right] = \prod_{j=1}^n (1-\lambda_j)^{-1/2}.$$

Since $d(1) = \prod_{j=1}^\infty (1-\lambda_j)$, the theorem readily follows from (3.10), (3.11), (3.12), and (3.8).

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