REMARKS ON WEAK COMPACTNESS OF OPERATORS DEFINED ON CERTAIN INJECTIVE TENSOR PRODUCTS

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ABSTRACT. We show that if X is a $\mathcal{L}_\infty$ -space with the Dieudonné property and Y is a Banach space not containing l_1 , then any operator $T: X \otimes_\varepsilon Y \rightarrow Z$, where Z is a weakly sequentially complete Banach space, is weakly compact. Some other results of the same kind are obtained.

Let X be a $\mathcal{L}_\infty$ -space (see [1] for this notion and some useful results on $\mathcal{L}_\infty$ -spaces) and Y be a Banach space not containing l_1 . We consider the injective tensor product $X \otimes_\varepsilon Y$ (see [3]), and we investigate the following problem: when is any operator $T: X \otimes_\varepsilon Y \rightarrow Z$, where Z is a Banach space, weakly compact?

In the case of $X = C(K)$ there are some papers devoted to the study of this question (see [2, 6–9]), but nothing seems to be known in the present setting; we observe that the theorems proved in the paper extend all of the above-quoted results, but their proofs make use of the results in [2, 8], so that they may be considered interesting complements to those theorems. Because the proofs of our results are similar, we give the proof of Theorem 2 only and leave the others to the reader. We need the following definition: *a Banach space E has the Dieudonné property if any weakly completely continuous (or Dieudonné) operator defined on it is weakly compact* [8].

We recall that $C(K)$ spaces have the Dieudonné property.

Lemma 1. $X^{**} \otimes_\varepsilon Y$ is a closed subspace of $(X \otimes_\varepsilon Y)^{**}$.

Proof. Let $x^{**} \otimes y$ be an element of $X \otimes Y$ and consider $\varphi \in (X \otimes_\varepsilon Y)^* = B^\pi(X, Y)$ (see [3] for a definition of $B^\pi(X, Y)$). Since $B^\pi(X, Y)$ is a closed subspace of $B^\pi(X^{**}, Y) = (X^{**} \otimes_\varepsilon Y)^*$ (see [3]) and since $\|\varphi\|_{B^\pi(X, Y)} = \|\varphi\|_{B^\pi(X^{**}, Y)}$ for any $\varphi \in B^\pi(X, Y)$, we have $|\varphi(x^{**} \otimes y)| \leq \|\varphi\|_\pi \|x^{**} \otimes y\|_\varepsilon$ and so $x^{**} \otimes y \in (X \otimes_\varepsilon Y)^{**}$; hence $\sum_{i=1}^p x_i^{**} \otimes y_i \in (X \otimes_\varepsilon Y)^{**}$. Now, we have to show that $\|\sum_{i=1}^p x_i^{**} \otimes y_i\|_\varepsilon = \|\sum_{i=1}^p x_i^{**} \otimes y_i\|_{(X \otimes_\varepsilon Y)^{**}}$. The inequality $\|\sum_{i=1}^p x_i^{**} \otimes y_i\|_\varepsilon \leq \|\sum_{i=1}^p x_i^{**} \otimes y_i\|_{(X \otimes_\varepsilon Y)^{**}}$ follows very easily from the very definition of ε -norm [3], the weak* density of B_{X^*} in $B_{X^{**}}$, and the fact that

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any $x^* \otimes y^* \in B^\pi(X, Y)$. The reverse inequality follows since we have

$$\left\| \sum_{i=1}^p x_i^{**} \otimes y_i \right\|_{(X \otimes_\varepsilon Y)^{**}} = \sup \left\{ \left\| \left(\sum_{i=1}^p x_i^{**} \otimes y_i \right) (\varphi) \right\| : \varphi \in B^\pi(X, Y), \|\varphi\|_\pi \leq 1 \right\},$$

any $\varphi \in B^\pi(X, Y)$ actually is an element of $B^\pi(X^{**}, Y)$, and this inclusion is an isometry [3], whereas we also have

$$\left\| \sum_{i=1}^p x_i^{**} \otimes y_i \right\|_\varepsilon = \sup \left\{ \left\| \left(\sum_{i=1}^p x_i^{**} \otimes y_i \right) (\varphi) \right\| : \varphi \in B^\pi(X^{**}, Y), \|\varphi\|_\pi \leq 1 \right\}.$$

Because the elements of the type $\sum_{i=1}^p x_i^{**} \otimes y_i$ are dense in $X^{**} \otimes_\varepsilon Y$, we are done.

We are now ready for the first theorem of the paper.

Theorem 2. *Let X be a $\mathcal{L}_\infty$ -space with the Dieudonné property and Y be a Banach space not containing l_1 . If Z is a weakly sequentially complete Banach space, then any $T: X \otimes_\varepsilon Y \rightarrow Z$ is weakly compact.*

Proof. Let us consider $T^{**}: (X \otimes_\varepsilon Y)^{**} \rightarrow Z^{**}$ and its restriction $\tilde{T}$ to all of $X^{**} \otimes_\varepsilon Y$ (which contains $X \otimes_\varepsilon Y$ as a closed subspace [3]). $\tilde{T}$ is continuous, because $X^{**} \otimes_\varepsilon Y$ is a closed subspace of $(X \otimes_\varepsilon Y)^{**}$, by virtue of Lemma 1.

We want to prove that $\tilde{T}$ takes its values in Z . Consider $x^{**} \otimes y \in B_{X^{**} \otimes_\varepsilon Y}$ and suppose (as we may) that $\|x^{**}\| \leq 1$. Then there is a net $(x_\alpha) \subset B_X$ such that $x_\alpha \rightarrow^{w^*} x^{**}$. Of course, we have

$$w^* - \lim_\alpha \tilde{T}(x_\alpha \otimes y) = \tilde{T}(x^{**} \otimes y).$$

Now define $\tilde{T}_y: X \rightarrow Z$ by putting $\tilde{T}_y(x) = T(x \otimes y)$. $\tilde{T}_y$ is a Dieudonné operator since Z is weakly sequentially complete; and so $\tilde{T}_y$ is weakly compact, since X has the Dieudonné property. Hence $(\tilde{T}_y(x_\alpha))$ is a relatively weakly compact subset of Z ; because $\tilde{T}_y(x_\alpha) = T(x_\alpha \otimes y) = \tilde{T}(x_\alpha \otimes y)$ for all α , the weak closure of the net $\tilde{T}(x_\alpha \otimes y)$ is in Z and so even any weak* cluster point of $\tilde{T}(x_\alpha \otimes y)$ must lie in Z ; hence $\tilde{T}(x^{**} \otimes y) \in Z$. This means that $\tilde{T}(\sum_{i=1}^p x_i^{**} \otimes y_i) \in Z$ for any $\sum_{i=1}^p x_i^{**} \otimes y_i \in X^{**} \otimes_\varepsilon Y$. The density of the elements of the type $\sum_{i=1}^p x_i^{**} \otimes y_i$ in $X^{**} \otimes_\varepsilon Y$ and the continuity of $\tilde{T}$ on $X^{**} \otimes_\varepsilon Y$ give our claim: $\tilde{T}$ takes its values in Z .

Now recall that X^{**} has the metric approximation property (see [3]) and so $X^{**} \otimes_\varepsilon Y = K_{w^*}(Y^*, X^{**})$ (= the Banach space of all $w^* - w$ continuous compact operators from Y^* into X^{**}); furthermore, X^{**} is complemented in some $C(K)$ space [1] and so $K_{w^*}(Y^*, X^{**})$ is complemented in $C(K, Y)$ by a projection P . The operator $\tilde{T} \circ P: C(K, Y) \rightarrow Z$ is a Dieudonné operator that must be weakly compact because of the result of [8]. Since $\tilde{T} \circ P$ restricted to $X \otimes_\varepsilon Y$ is nothing else than T , we are done.

A similar proof (making use of the main result of the paper [2]) gives the following theorem (for the definitions of Pelczynski's properties (V) and (u) we refer to [2]).

Theorem 3. *Let X be a $\mathcal{L}_\infty$ -space with Pelczynski's property (V) and Y be a Banach space with Pelczynski's property (u), not containing l_1 . If Z is a Banach space not containing c_0 , then any $T: X \otimes_\varepsilon Y \rightarrow Z$ is weakly compact.*

Theorem 2 has the following corollary about a new isomorphic property recently considered by Saab and Saab in [10]: a Banach space E is said to possess property (w) if any operator $T: E \rightarrow E^*$ is weakly compact.

Corollary 4. *Let X be a $\mathcal{L}_\infty$ -space with Dieudonné property and Y be a Banach space not containing l_1 such that Y^* is weakly sequentially complete. Then $X \otimes_\varepsilon Y$ has property (w).*

Proof. It will be enough to show that $(X \otimes_\varepsilon Y)^*$ is weakly sequentially complete, because in such a case we can apply Theorem 2. $B^\pi(X, Y) = (X \otimes_\varepsilon Y)^*$ is a closed subspace of $B^\pi(X^{**}, Y) = (X^{**} \otimes_\varepsilon Y)^*$ (see [3]); since $X^{**} \otimes_\varepsilon Y$ is complemented in some $C(K, Y)$ space (see the proof of Theorem 2), $B^\pi(X^{**}, Y)$ is complemented in $(C(K, Y))^*$. But now it is well known that this last space is weakly sequentially complete (see, e.g., [10, Proposition 5]). We are done. Corollary 4 is an improvement of Proposition 5 in [10].

At the end, we recall that a Banach space E has the Dunford-Pettis property if any weakly compact operator on E is a Dunford-Pettis (or completely continuous) operator. We observe that any $\mathcal{L}_\infty$ -space enjoys the Dunford-Pettis property [1]. Using the same technique employed in Theorem 2, we have that $X \otimes_\varepsilon Y$ has the Dunford-Pettis property whenever X is a $\mathcal{L}_\infty$ -space and Y is a Banach space with the Dunford-Pettis property such that $C(X, Y)$ has the same property, for any Hausdorff compact space K . In particular, the result in [5] and results by Bourgain quoted in [4] imply that we can choose Y with the Schur property or $Y = L^1(\mu)$; in this last case $X \otimes_\varepsilon Y$ is isomorphic to the completion of the space of Pettis integrable functions (see [3]). More generally we can show that the injective tensor product of a $\mathcal{L}_\infty$ -space and a $\mathcal{L}_1$ -space has the Dunford-Pettis property.

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