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ON THE MAPPING CLASS GROUP OF SPHERICAL 3-ORBIFOLDS

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ABSTRACT. We prove that the mapping class group of a spherical 3-orbifold with nonempty singular set is finite and can be realized by a finite group of diffeomorphisms; we also indicate how to compute this group.

1. INTRODUCTION

Let G be a finite group of orientation-preserving diffeomorphisms of the 3-sphere S^3, and let $\mathscr{O} := S^3/G$, a closed orientable 3-orbifold. We will always assume in this paper that the singular set Σ of $\mathscr{O}$ is nonempty (the projection of the fixed-point sets of the nontrivial elements in G). Let $M := \mathscr{O} - \overset{\circ}{N}(\Sigma)$, where $\overset{\circ}{N}(\Sigma)$ denotes the interior of a regular neighbourhood of Σ, so M is a compact orientable 3-manifold with nonempty boundary. By [GL, Theorem 2], M is irreducible and atoroidal (because S^3 is so). By [T1, Corollary 2.4], we have the following cases:

(I) M is a Seifert fiber space, or

(II) M is hyperbolic; i.e., its interior $\overset{\circ}{M}$ admits a complete hyperbolic structure (but not Seifert fibered: manifolds like the solid torus and the product of torus and interval we count to the first class so that the two classes become disjoint).

Theorem 1. *The mapping class group $\pi_0 \operatorname{Diff}(\mathscr{O})$ of diffeomorphisms of $\mathscr{O}$ modulo isotopy is finite and can be realized by a finite group of diffeomorphisms of $\mathscr{O}$, i.e., there exists a finite group of diffeomorphisms of $\mathscr{O}$ (isomorphic to $\pi_0 \operatorname{Diff}(\mathscr{O})$ if M is hyperbolic) that projects onto $\pi_0 \operatorname{Diff}(\mathscr{O})$ under the canonical map $\operatorname{Diff}(\mathscr{O}) \to \pi_0 \operatorname{Diff}(\mathscr{O})$.*

Here the terms diffeomorphism and isotopy are used in the orbifold sense, i.e., respecting the singular set; see [BS, DM, D1, D2, Ta] for basic facts about orbifolds.

By restricting and extending diffeomorphisms and isotopies, we get an isomorphism of mapping class groups $\pi_0 \operatorname{Diff}(\mathscr{O}) \cong \pi_0 \operatorname{Diff}_e(M)$, where $\operatorname{Diff}_e(M)$ denotes the subgroup of diffeomorphisms of M that extend to $\mathscr{O}$. If M is Seifert fibered (of one of a few very special cases, see the proof of Theorem 1), a

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computation of $\pi_0 \operatorname{Diff}(M)$, and then also of $\pi_0 \operatorname{Diff}_e(M)$, is more or less well known (see [Jo, Proposition 2.5.3]), so we concentrate here on the more interesting case where M is hyperbolic. We also will assume now that $G \subset SO(4)$, i.e., that G operates by orthogonal maps; by [T1, T2], every finite group of orientation-preserving diffeomorphisms of S^3 such that some nontrivial element has fixed points is conjugate to an orthogonal group, so this is really no restriction (but it is not needed in the proof of Theorem 1).

Theorem 2. *Let $G \subset SO(4)$ be finite such that the spherical 3-orbifold $\mathscr{O} = S^3/G$ has nonempty singular set Σ and $M = S^3 - \mathring{N}(\Sigma)$ is hyperbolic. Then we have isomorphisms $\pi_0 \operatorname{Diff}(\mathscr{O}) \cong NG/G \cong \operatorname{Isom}(\mathscr{O})$, where NG denotes the normalizer of G in the orthogonal group $O(4)$ and $\operatorname{Isom}(\mathscr{O})$ the group of isometries of the spherical 3-orbifold $\mathscr{O}$; in particular, $\pi_0 \operatorname{Diff}(\mathscr{O})$ can be realized by isometries.*

Of course, the isomorphism $NG/G \cong \operatorname{Isom}(\mathscr{O})$ is immediate, so we have to prove only $\pi_0 \operatorname{Diff}(\mathscr{O}) \cong NG/G$.

Remarks. (a) Explicit computations in the interesting special case where $\mathscr{O}$ is S^3 with a spherical Montesinos link as singular set are given in [Sa].

(b) A list of all spherical 3-orbifolds with S^3 as underlying topological space can be found in [D2].

(c) For the mapping class groups of spherical 3-manifolds see [HR, BO] and the references given there.

2. Proof of Theorem 1

We have the isomorphism $\pi_0 \operatorname{Diff}(\mathscr{O}) \cong \pi_0 \operatorname{Diff}_e(M)$. Let $p: S^3 \to \mathscr{O} = S^3/G$ be the projection and $\tilde{\Sigma} := p^{-1}(\Sigma)$, the union of the fixed point sets of the nontrivial elements in G. We consider the two cases M Seifert fibered and M hyperbolic.

(I) *M is Seifert fibered.* Then the boundary ∂M of M consists of tori and Σ and $\tilde{\Sigma}$ consist of disjoint circles, so $\tilde{\Sigma}$ is a link in S^3 whose complement $S^3 - \tilde{\Sigma}$ is Seifert fibered. Such links have been classified in [BM]: they are either (isotopic to) a sublink of a Seifert fibration of S^3 (i.e., a collection of fibers) or of the form in Figure 1. If $n > 1$ (see Figure 1), then G maps Σ_0 to itself, so G is a finite cyclic group or a direct sum of two finite cyclic groups. But this would imply $n \leq 1$. It follows that in any case the Seifert fibration of $S^3 - \tilde{\Sigma}$ extends to a Seifert fibration of S^3 invariant under the action of G.

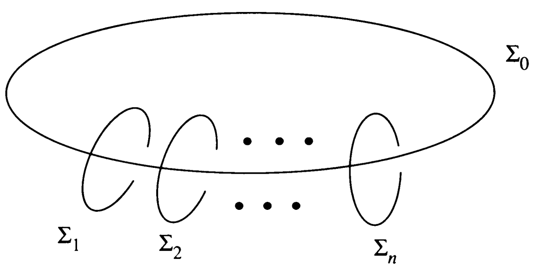


FIGURE 1

In particular, $\mathscr{O}$ is a Seifert fibered 3-orbifold with fundamental group G (see [DM, BS]). Let B be the base of the Seifert fibration of $\mathscr{O}$, and let S be the union of Σ and all exceptional fibers of $\mathscr{O}$ (that are not in Σ). Because the orbifold fundamental group $\pi_1\mathscr{O} \cong G$ is finite, B is the 2-sphere or real projective space and S has at most three components. Then also the Seifert fiber space $M = \mathscr{O} - \mathring{N}(\Sigma)$ is of a very special form; in particular, the mapping class group of the base B of the Seifert fibration of M is a finite group. The mapping class groups of Seifert fiber spaces have been computed in [Jo, §25], excluding a few special cases that can be done directly. Using the fact that the elements of $\pi_0 \operatorname{Diff}_e(M)$ preserve, up to isotopy and orientation, some nontrivial simple closed curve on each boundary component of M and that the mapping class group of the base is finite, it is easy to see that in each of the possible cases the mapping class group $\pi_0 \operatorname{Diff}_e(M)$ of extendable diffeomorphisms is finite. Moreover, by the solution of the Nielsen realization problem for Seifert fiber spaces [Z1, Z2], there exists a finite group of diffeomorphisms of M projecting onto $\pi_0 \operatorname{Diff}_e(M)$ (there might be an obstruction to realizing $\pi_0 \operatorname{Diff}_e(M)$ by an isomorphic group of representing diffeomorphisms; we do not know if this obstruction really occurs in the very special cases at hand). Extending this finite group of diffeomorphisms from M to $\mathscr{O}$, we finish the proof of Theorem 1 in the first case. The second case is

(II) *M is hyperbolic.* This divides naturally into two subcases:

(a) *M has finite volume.* In this case, $\pi_0 \operatorname{Diff}_e(M) \subset \pi_0 \operatorname{Diff}(M)$ are finite groups: by [Wa, Corollary 7.5], $\pi_0 \operatorname{Diff}(M) \cong \operatorname{Out} \pi_1 M$ (the outer automorphism group). As a consequence of Mostow's rigidity theorem $\operatorname{Out} \pi_1 M$ is a finite group isomorphic to the group of isometries of M (see [T3, Corollary 5.7.4]). Extending to $\mathscr{O}$ we get the result.

(b) *M has infinite volume.* In this case, Σ does not consist of circles only but also contains branching points of the form   , so Σ is a trivalent graph. Accordingly, the boundary ∂M of M is built out of pieces of the following form:

(i) tori that come from S^1-components of Σ;

(ii) spheres with three holes coming from points in Σ where three branches meet;

(iii) annuli connecting the pieces of type (ii).

Let $\partial_0 M$ consist of all pieces of type (ii), and let $D_0 M$ be the double of M along $\partial_0 M$. It has been shown in [Z3, proof of Theorem 2′] that $D_0 M$ is irreducible and atoroidal, so by [T1] we have again the two subcases $D_0 M$ hyperbolic and $D_0 M$ Seifert fibered.

(b1) *$D_0 M$ is hyperbolic* (of finite volume because the boundary of $D_0 M$ consists of tori only). By also doubling diffeomorphisms, $\pi_0 \operatorname{Diff}_e(M)$ extends to an isomorphic group of mapping classes of $D_0 M$ to which we adjoin the mapping class of the involution τ (reflection along $\partial_0 M \subset D_0 M$) that interchanges the two copies of M in $D_0 M$ to get a group of mapping classes $K \cong \pi_0 \operatorname{Diff}_e(M) \oplus \mathbb{Z}_2$ of $D_0 M$. As above, because $D_0 M$ has finite volume, K is finite and can be realized by an isomorphic group K' of isometries of $D_0 M$. By [To, Theorem B], we can assume $\tau \in K'$ (any two realizations by involutions of the mapping class of τ will be conjugate by a diffeomorphism of $D_0 M$). But then the realization of $\pi_0 \operatorname{Diff}_e(M) \subset K$ will map $\partial_0 M$ and both copies of M to itself (because it commutes with τ and $\partial_0 M$ is the fixed point

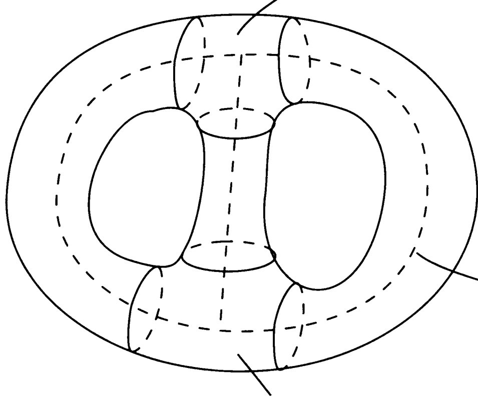


FIGURE 2

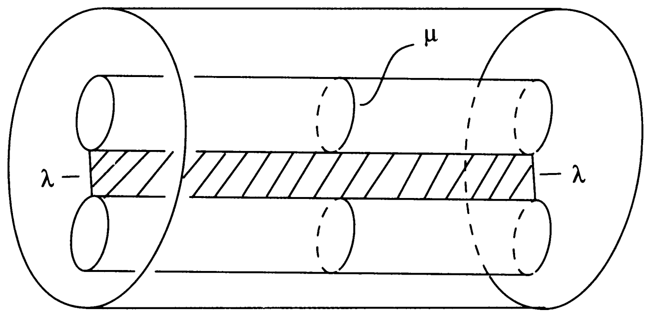


FIGURE 3

set of τ), so it can be restricted to M realizing $\pi_0 \operatorname{Diff}_e(M)$. Extending this group to $\mathscr{O}$, we get our result.

(b2) $D_0 M$ *is Seifert fibered*. By [Ja, Theorem VI.34], each component of $\partial_0 M \subset D_0 M$ (a sphere with three boundary components) is isotopic to a surface transverse to the Seifert fibration of $D_0 M$ and is a fiber of a fibration of $D_0 M$ as a surface bundle over S^1. Therefore, $\partial_0 M$ (which separates $D_0 M$ into two components) has exactly two components and M is homeomorphic to the product $T \times [0, 1]$, where T is the 2-sphere with three boundary components (so M is homeomorphic to the handlebody of genus 2). Moreover, $\pi_0 \operatorname{Diff}_e(M)$ preserves $T \times \{0, 1\}$, up to isotopy; in this case, Σ is the "θ-graph"; see Figure 2.

Because the mapping class group of T is finite, every element of a subgroup of finite index in $\pi_0 \operatorname{Diff}_e(M)$ has a representative f that is the identity restricted to $T \times \{0\}$ and $T \times \{1\}$. Then f restricted to any of the annuli in $\partial T \times [0, 1]$ is isotopic to a Dehn twist around the central curve μ of the annulus; see Figure 3.

Consider a simple closed curve λ in ∂M that traverses the annulus once and bounds a disk in M; see Figure 3. If the above Dehn twist is nontrivial, then λ is mapped by f to a curve in ∂M that does not bound a disk in M, which is impossible. Therefore, f is isotopic to the identity and $\pi_0 \operatorname{Diff}_e(M)$ is finite. It is also realizable by an isomorphic group of diffeomorphisms of M preserving its product structure because the mapping class group of T is realizable in such a way. Extending to $\mathscr{O}$, we get the result. This finishes the proof of Theorem 1.

3. Proof of Theorem 2

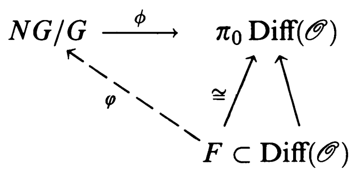
We need the following

Lemma. *Under the hypothesis of Theorem* 2, *the normalizer* NG *of* G *in* $\mathrm{O}(4)$ *is finite.*

Proof. The normalizer $NG = \overline{NG}$ is a closed Lie subgroup of $\mathrm{O}(4)$. If NG is not finite, i.e., not discrete, a circle subgroup S^1 of NG projects to a nontrivial circle action on $\mathscr{O}$ and on M. But for a hyperbolic 3-manifold that is not Seifert fibered, such a circle action does not exist.

We come to the proof of Theorem 2 now.

By Theorem 1, $\pi_0 \operatorname{Diff}(\mathscr{O})$ is finite and can be realized by an isomorphic group F of diffeomorphisms of $\mathscr{O}$, so we have the following diagram:



The map ϕ is the canonical map induced by projecting elements of NG to $\mathscr{O} = S^3/G$.

We want to define φ in the above diagram such that it becomes commutative. For this we note that, by [T2], the group $\widetilde{F}$ of all lifts of elements of F to S^3, which contains G as a normal subgroup, is conjugate to a subgroup of $\mathrm{O}(4)$, i.e., $f\widetilde{F}f^{-1} \subset \mathrm{O}(4)$, for some diffeomorphism of S^3 (because some nontrivial element of G, and hence of $\widetilde{F}$, has nonempty fixed point set, so [T2] applies). Then also $fGf^{-1} \subset \mathrm{O}(4)$; by a result of DeRham (see [Ro]), subgroups of $\mathrm{O}(4)$ that are conjugate by a diffeomorphism are also conjugate by an orthogonal map (this is true in any dimension; in dimension 3, it can also be proved by looking at the explicit classification of finite subgroups of $\mathrm{SO}(4)$ in [TS]). Therefore, we can assume that $\widetilde{F}$ is a subgroup of $\mathrm{O}(4)$ containing the original G, i.e., $\widetilde{F} \subset NG$. The map φ is now the obvious one.

The existence of φ implies that ϕ is surjective. Let d be an element in the kernel of ϕ. By the Lemma, the group NG/G that is clearly the group of isometries of $\mathscr{O}$ is finite, so we can assume $d^n = \mathrm{id}_{\mathscr{O}}$, where we consider d as an isometry of $\mathscr{O} = S^3/G$; also, d is isotopic to the identity of $\mathscr{O}$. But then the restriction of d to M also has finite order and is isotopic to the identity. Because M is hyperbolic and not Seifert fibered, the center of its fundamental group $\pi_1 M$ is trivial and the only such map is the identity (see [Co]), so $d = \mathrm{id}_{\mathscr{O}}$ and ϕ is also injective.

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