SPECTRAL INCLUSION RELATIONS FOR T , $T|Y$, AND T/Y

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(Communicated by Palle E. T. Jorgensen)

Dedicated to Professor T. Ando on his sixtieth birthday

ABSTRACT. Let X be a complex Banach space and Y be an invariant subspace of a bounded linear operator T in X . Then it is easy to prove that (1) $\sigma(T) \subset \sigma(T|Y) \cup \sigma(T/Y)$, (2) $\sigma(T|Y) \subset \sigma(T) \cup \sigma(T/Y)$, and (3) $\sigma(T/Y) \subset \sigma(T) \cup \sigma(T|Y)$. In this paper, we first study these relations for unbounded linear operators: While (3) holds without any conditions, (1) and (2) do not hold in general. We shall make clear conditions on T that guarantee (1) and (2). Next we introduce the notion of extended spectrum for unbounded linear operators and prove similar results for the extended spectrum.

1. PRELIMINARIES

In this paper, we deal with not necessarily closed unbounded linear operators. Since there are some variations for the usage of terminologies in this field, we begin by fixing our notations.

Let T be a linear operator on a complex Banach space X with domain $D(T)$. A closed subspace Y of X is said to be invariant under T if $T(D(T) \cap Y) \subset Y$. Let $\text{Lat}(T)$ be the family of all closed subspaces which are invariant under T . For $Y \in \text{Lat}(T)$, we denote by $T|Y$ the restriction of T to Y with domain $D(T|Y) = Y \cap D(T)$ and by T/Y the quotient operator induced by T on X/Y with domain $D(T/Y) = \{\hat{x} \in X/Y \mid x \in D(T)\}$, where $\hat{x}$ denotes the coset including $x \in X$.

The resolvent set of T is defined as the set $\rho(T)$ of all $\lambda \in \mathbb{C}$ such that the range of $\lambda I - T$ is dense in X and $\lambda I - T$ has a continuous inverse, i.e., there exists $M > 0$ such that $M\|x\| \leq \|(\lambda I - T)x\|$ for all $x \in D(T)$. The spectrum of T is defined by $\sigma(T) = \mathbb{C} \setminus \rho(T)$ as usual. Note that $\sigma(\bar{T}) = \sigma(T)$ for the closure $\bar{T}$ of a closable operator T . We refer to [4] for details.

The set of all bounded (resp. closed) linear operators on X is denoted by $B(X)$ (resp. $C(X)$).

Received by the editors June 25, 1990 and, in revised form, April 5, 1991.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 47A10.

Key words and phrases. Spectrum, extended spectrum.

The authors would like to express their gratitude to Professor T. Yoshino and the referee for their helpful comments.

2. SPECTRAL INCLUSION RELATIONS FOR $\sigma(T)$, $\sigma(T|Y)$, AND $\sigma(T/Y)$

The following result is fundamental in the spectral inclusion relations (cf. Erdelyi and Lange [1, Proposition 1.14]).

Proposition 2.1. *Let $T \in B(X)$ and $Y \in \text{Lat}(T)$. Then the following relations hold.*

- (1) $\sigma(T) \subset \sigma(T|Y) \cup \sigma(T/Y).$
- (2) $\sigma(T|Y) \subset \sigma(T) \cup \sigma(T/Y).$
- (3) $\sigma(T/Y) \subset \sigma(T) \cup \sigma(T|Y).$

Nagy [2] proved that if T and T/Y are closed operators, then all of the above relations hold. In the following, we consider a not necessarily closed operator T on X . Our first result is that the relation (3) holds without any other conditions.

Theorem 2.2. *Let $T : X \supset D(T) \rightarrow X$ be a linear operator and $Y \in \text{Lat}(T)$. Then*

$$\sigma(T/Y) \subset \sigma(T) \cup \sigma(T|Y).$$

Proof. We prove that $\rho(T) \cap \rho(T|Y) \subset \rho(T/Y)$. Let $\lambda \in \rho(T) \cap \rho(T|Y)$. We have to show that λ is in the resolvent set of T/Y . The density of the range of $\lambda I - T/Y$ follows from that of $\lambda I - T$. To see that $\lambda I - T/Y$ has a continuous inverse, we first note that there exists $M > 0$ such that

$$M\|z\| \leq \|(\lambda I - T)z\|, \quad \forall z \in D(T)$$

(because $\lambda \in \rho(T)$). Let $x \in D(T)$ and $\hat{z} = (\lambda I - T/Y)\hat{x}$. Then $(\lambda I - T)x - z \in Y$. Since $\lambda I - T|Y$ has a dense range (because $\lambda \in \rho(T|Y)$), for any $y \in Y$ we can find a sequence $\{y_n\}_{n \geq 1}$ in $Y \cap D(T)$ such that

$$\lim_{n \rightarrow \infty} (\lambda I - T|Y)y_n = y + z - (\lambda I - T)x.$$

Now using the above inequality, we obtain

$$M\|\hat{x}\| \leq \lim M\|x + y_n\| \leq \lim \|(\lambda I - T)(x + y_n)\| = \|y + z\|$$

Since $y \in Y$ is arbitrary, we have $M\|\hat{x}\| \leq \|\hat{z}\|$, which implies that $\lambda I - T/Y$ has a continuous inverse. Thus $\lambda \in \rho(T/Y)$. $\square$

The following theorem shows that the relation (2) holds if T is closed.

Theorem 2.3. *Let $T \in C(X)$ and $Y \in \text{Lat}(T)$. Then*

$$\sigma(T|Y) \subset \sigma(T) \cup \sigma(T/Y).$$

Proof. We prove that $\rho(T) \cap \rho(T/Y) \subset \rho(T|Y)$. Let $\lambda \in \rho(T) \cap \rho(T/Y)$. Since $T|Y \in C(Y)$, we have to prove that $\lambda I - T|Y$ is bijective. Since $\lambda I - T$ is injective, the injectivity of $\lambda I - T|Y$ is clear. To see the surjectivity, take $y \in Y$. Since $\lambda I - T$ is surjective, there exists $x \in D(T)$ such that $(\lambda I - T)x = y$. Then $(\lambda I - T/Y)\hat{x} = \hat{y} = \hat{0}$ and hence $\hat{x} = \hat{0}$. Thus $x \in Y$ and $(\lambda I - T|Y)x = y$. $\square$

The following example shows that the relation (2) breaks down for a non-closed operator.

Example 2.4. Let $X = L^2[0, 1]$ and D be the set of all polynomials in X . Let T be an operator with domain D defined by $Tx = 0$ for $x \in D$. Then $\overline{T} = 0$ and $\sigma(T) = \{0\}$. If we take $Y = \{ae^t \mid a \in \mathbb{C}\}$, then $Y \in \text{Lat}(T)$ and $D(T|Y) = D \cap Y = \{0\}$. Hence $\sigma(T|Y) = \mathbb{C}$. Also $\overline{T/Y} = 0/Y$ and $\sigma(T/Y) = \{0\}$. Thus the relation (2) does not hold in this case.

The next example was considered by Nagy in [3], where he proved that $T \in C(X)$ but T/Y is not closable for some $Y \in \text{Lat}(T)$. We prove that the relation (1) does not hold for this T and a certain T -invariant subspace.

Example 2.5. Consider a Banach space X defined by

$$X = \left\{ x = (a_0, a_1, a_2, \dots; b_1, b_2, b_3, \dots) \mid \|x\| = \sum_{k=0}^{\infty} |a_k| + \sum_{j=1}^{\infty} |b_j| < \infty \right\}$$

and a dense linear subspace D defined by

$$D = \left\{ x \in X \mid \sum_{k=1}^{\infty} k|a_k| < \infty \right\}.$$

Let T be an operator with domain D defined by

$$Tx = \left(\sum_{k=1}^{\infty} ka_k, 0, 0, \dots; a_1 + b_1, 2a_2 + b_2, 3a_3 + b_3, \dots \right) \quad \text{for } x \in D.$$

Set

$$Y = \{x \in X \mid 0 = a_0 = a_1 = \dots\}, \quad Z = \{x \in X \mid 0 = a_1 = a_2 = \dots\}.$$

Then $Y, Z \in \text{Lat}(T)$ and $Z \subset D$. Nagy [3] proved that $T \in C(X)$ but T/Y is not closable. On the other hand, $T|Z \in B(Z)$ and therefore $\sigma(T|Z)$ is bounded. It is easy to check that $0 \in \sigma(T)$ (in fact it is an eigenvalue of T). For $\lambda \neq 0$, there is no $x \in D$ such that

$$(\lambda - T)x = (0, 1^{-2}, 2^{-2}, 3^{-2}, \dots; 0, 0, \dots) \in X,$$

which implies $\sigma(T) = \mathbb{C}$. Since $D(T/Z)$ is dense in X and $\overline{T/Z} = 0$ on this subspace, the closure of T/Z is the zero operator in $B(X/Y)$ and hence $\sigma(T/Z) = \{0\}$. Thus the relation (1) does not hold for T and Z .

3. SPECTRAL INCLUSION THEOREMS FOR $\sigma_e(T)$, $\sigma_e(T|Y)$, AND $\sigma_e(T/Y)$

In this section, we study spectral inclusion relations for the extended spectrum $\sigma_e(T)$, $\sigma_e(T|Y)$, and $\sigma_e(T/Y)$.

Definition 3.1. Let $T \in C(X)$. The extended spectrum of T is a subset $\sigma_e(T)$ in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ defined by

$$\sigma_e(T) = \begin{cases} \sigma(T) & \text{if } T \in B(X), \\ \sigma(T) \cup \{\infty\} & \text{otherwise.} \end{cases}$$

For $T \in C(X)$, it is proved that the set $\mathbb{C}_\infty \setminus \sigma_e(T)$ is the domain of the analytic function $(\lambda I - T)^{-1}$. We shall extend the above definition of extended spectrum for closed operators to all linear operators on X .

Let T be a linear operator on X . For $\lambda \in \rho(T)$, $(\lambda I - T)^{-1}$ is a densely defined continuous operator, and hence it has a unique extension $A_\lambda \in B(X)$. Since $A_\lambda - A_\mu = (\mu - \lambda)A_\lambda A_\mu$ for $\lambda, \mu \in \rho(T)$, it follows that A_λ is analytic on $\rho(T)$ (see [4] for details). Thus the following definition is natural.

Definition 3.2. Let $T : X \supset D(T) \rightarrow X$ be an operator. The extended spectrum $\sigma_e(T)$ of T is defined to be

$$\begin{cases} \sigma(T) & \text{if } \sigma(T) \text{ is bounded and } A_\lambda \text{ is analytic at } \infty, \\ \sigma(T) \cup \{\infty\} & \text{otherwise.} \end{cases}$$

The following two theorems show that the same inclusion relations hold for the extended spectrum as for the usual one.

Theorem 3.3. Let $T : X \supset D(T) \rightarrow X$ be an operator and $Y \in \text{Lat}(T)$. Then $\sigma_e(T/Y) \subset \sigma_e(T) \cup \sigma_e(T|Y)$.

Proof. Due to Theorem 2.2, we have to prove that if $\infty \notin \sigma_e(T) \cup \sigma_e(T|Y)$, then $\infty \notin \sigma_e(T/Y)$. Suppose that $\infty \notin \sigma_e(T) \cup \sigma_e(T|Y)$. From the definition, $\sigma(T)$ and $\sigma(T|Y)$ are bounded, and so is $\sigma(T/Y)$ by Theorem 2.2. Thus if $|\lambda|$ is sufficiently large, $(\lambda I - T)^{-1}$, $(\lambda I - T|Y)^{-1}$, and $(\lambda I - T/Y)^{-1}$ have continuous extensions in $B(X)$, which are denoted by A_λ , B_λ , and C_λ . By the assumption, A_λ and B_λ are analytic at ∞ .

Now we prove that C_λ is analytic at ∞ by several steps.

(i) $Y \in \text{Lat}((\lambda I - T)^{-1})$: Let $y \in D((\lambda I - T)^{-1}) \cap Y = R(\lambda I - T) \cap Y$ and set $x = (\lambda I - T)^{-1}y \in X$. Then $y = (\lambda I - T)x \in Y$. Since the range of $\lambda I - T|Y$ is dense in Y , there exists a sequence $\{y_n\}_{n \geq 1} \subset D(T|Y)$ such that $\lim_{n \rightarrow \infty} (\lambda I - T|Y)y_n = y$. By the definition of the resolvent set, one can find $M > 0$ such that

$$M\|y_n - x\| \leq \|(\lambda I - T)(y_n - x)\|, \quad \forall n \geq 1,$$

from which $x = \lim y_n \in Y$ and Y is invariant under $(\lambda I - T)^{-1}$.

(ii) $Y \in \text{Lat}(A_\lambda)$: Let $y \in Y$. Since the range of $\lambda I - T|Y$ is dense in Y , there exists a sequence $\{y_n\} \subset D(T|Y)$ such that $\lim_{n \rightarrow \infty} (\lambda I - T|Y)y_n = y$. Then

$$M\|y_n - y_m\| \leq \|(\lambda I - T)(y_n - y_m)\| \rightarrow 0 \quad (n, m \rightarrow \infty).$$

Hence there exists $y_0 \in Y$ such that $\lim y_n = y_0$. Let $z_n = (\lambda I - T|Y)y_n = (\lambda I - T)y_n \in Y$. Then $(\lambda I - T)^{-1}z_n = y_n \rightarrow y_0$ ($n \rightarrow \infty$) and $z_n \rightarrow y_0$ ($n \rightarrow \infty$), which shows that $A_\lambda y = y_0 \in Y$. Thus $Y \in \text{Lat}(A_\lambda)$.

(iii) $(\lambda I - T)^{-1}/Y = (\lambda I - T/Y)^{-1}$: Let $z \in D(T)$ and $x = (\lambda I - T)z \in X$. Then $(\lambda I - T/Y)\hat{z} = \hat{x}$ and $(\lambda I - T/Y)^{-1}\hat{x} = \hat{z}$. On the other hand, we have $(\lambda I - T)^{-1}x = z$ and hence $[(\lambda I - T)^{-1}/Y]\hat{x} = \hat{z}$. Thus $(\lambda I - T)^{-1}/Y \subset (\lambda I - T/Y)^{-1}$.

Conversely, take $\hat{x} \in D((\lambda I - T/Y)^{-1})$ and set $\hat{z} = [(\lambda I - T/Y)^{-1}]\hat{x}$ with $z \in D(T)$. Then $(\lambda I - T/Y)\hat{z} = \hat{x}$ and $(\lambda I - T)z - x \in Y$. If we set $y = (\lambda I - T)z - x$, then $x + y = (\lambda I - T)z$ and it is in the domain of $(\lambda I - T)^{-1}$ and hence $\hat{x} \in D((\lambda I - T)^{-1}/Y)$. Thus $(\lambda I - T)^{-1}/Y = (\lambda I - T/Y)^{-1}$.

(iv) $C_\lambda = A_\lambda/Y$: Since $(\lambda I - T/Y)^{-1} \subset C_\lambda \in B(X/Y)$ and $(\lambda I - T)^{-1}/Y \subset A_\lambda/Y$, this follows from the step (iii).

Now the analyticity of C_λ at ∞ follows from the analyticity of A_λ at ∞ . $\square$

Theorem 3.4. Let $T \in C(X)$ and $Y \in \text{Lat}(T)$. Then $\sigma_e(T|Y) \subset \sigma_e(T) \cup \sigma_e(T/Y)$.

Proof. If $\infty \notin \sigma_e(T) \cup \sigma_e(T/Y)$, $T \in B(X)$ because $T \in C(X)$. In particular, $T|Y \in B(Y)$ and $\infty \notin \sigma_e(T|Y)$. Now the above inclusion follows from Theorem 2.3. $\square$

The next two theorems show that if T is either densely defined or closable, then $\sigma_e(T)$ admits the same characterization as in the case of a closed operator.

Theorem 3.5. Let $T : X \supset D(T) \rightarrow X$ be an operator with dense domain $D(T)$. Then

$$\sigma_e(T) = \begin{cases} \sigma(T) & \text{if } T \text{ is closable and its closure } \bar{T} \text{ is in } B(X), \\ \sigma(T) \cup \{\infty\} & \text{otherwise.} \end{cases}$$

Proof. Let X' be the dual space of X and T' be the dual operator of T . Then $\rho(T) = \rho(T')$ and $((\lambda I - T)^{-1})' = (\lambda I - T')^{-1} \in B(X')$ for $\lambda \in \rho(T)$. If $\infty \notin \sigma_e(T)$, then the continuous extension A_λ of $(\lambda I - T)^{-1}$ in $B(X)$ is analytic at ∞ . Since $A'_\lambda \subset ((\lambda I - T)^{-1})' = (\lambda I - T')^{-1} \in B(X')$, it follows that $A'_\lambda = (\lambda I - T')^{-1}$, from which one sees that $(\lambda I - T')^{-1}$ is analytic at ∞ . Since T' is closed, this implies that $T' \in B(X')$. Now $|\langle Tx, x' \rangle| = |\langle x, T'x' \rangle| \leq \|x\| \|T'\| \|x'\|$ for $x \in D(T)$ and $x' \in X'$, which shows that T is closable and $\bar{T} \in B(X)$.

Conversely, suppose that T is closable and its closure $\bar{T}$ is in $B(X)$. Then the spectrum $\sigma(T) = \sigma(\bar{T})$ is bounded and $(\lambda I - \bar{T})^{-1}$ is analytic at ∞ . Since $(\lambda I - T)^{-1} \subset (\lambda I - \bar{T})^{-1} \in B(X)$ for $\lambda \in \rho(T)$, we conclude that $A_\lambda = (\lambda I - \bar{T})^{-1}$ and it is analytic at ∞ . $\square$

Theorem 3.6. Let T be a closable operator. Then

$$\sigma_e(T) = \begin{cases} \sigma(T) & \text{if } \bar{T} \in B(X), \\ \sigma(T) \cup \{\infty\} & \text{otherwise.} \end{cases}$$

Proof. If T is closable, then $A_\lambda = (\lambda I - \bar{T})^{-1}$ for $\lambda \in \rho(T)$. So if $\sigma(T)$ is bounded and A_λ is analytic at ∞ , then $\bar{T} \in B(X)$. Converse implication is also true from the argument in the proof of Theorem 3.5. $\square$

Theorem 3.6 shows that if T is closable and $\bar{T} \in B(X)$, then $\infty \notin \sigma_e(T)$. One might expect that the converse implication holds as well, however this is not the case as seen in the following example.

Example 3.7. Consider a Hilbert space $X = L^2[-\pi, \pi]$. Let $\{e_n\}_{n \in \mathbb{Z}}$ be a complete orthonormal system defined by $e_n(t) = e^{int}/2\pi$ for $t \in [-\pi, \pi]$, P be an orthogonal projection onto the subspace spanned by $\{e_n\}_{n \neq 1}$, and D be the set of all polynomials in X . Define an operator T with domain $D(T) = P(D)$ by $TPx = x$, for $x \in D$. (This is well defined because P is 1-1 on D .) Clearly $D(T)$ is not dense and T is not closable, because if one takes a sequence $x_n \in D$ that converges to e_1 , then $Px_n \rightarrow Pe_1 = 0$ ($n \rightarrow \infty$), but $TPx_n = x_n \rightarrow e_1 \neq 0$ ($n \rightarrow \infty$).

We claim that $\sigma(T) = \{1\}$. In fact, since $e_0 \in D$ and $TPe_0 = e_0 = Pe_0$, 1 is in the point spectrum of T . Conversely take a complex number $\lambda \neq 1$. We first show that $\lambda I - T$ is injective. Let $x \in D$ and suppose that $(\lambda I - T)Px = 0$. Then $x = \lambda Px$. Applying P on this equation, we have $Px = \lambda Px = \lambda P^2x$, which means that $Px = 0$ (because $\lambda \neq 1$), and hence $x = \lambda Px = 0$.

Next we prove that $(\lambda I - T)P(D)$ is dense in X . To see this, let $y \in X$ and suppose that $\langle (\lambda I - T)Px, y \rangle = 0$ for all $x \in D$. Then $\langle x, y \rangle = \langle \lambda Px, y \rangle = \langle x, \bar{\lambda}Py \rangle$, from which $y = \bar{\lambda}Py$ since D is dense in X . Now the above reasoning in the proof of the injectivity of $\lambda I - T$ shows that $y = 0$, proving the density of $(\lambda I - T)P(D)$.

Finally we prove that $\lambda I - T$ has a continuous inverse. For $x \in D$,

$$\begin{aligned} \|(\lambda I - T)Px\|^2 &= \langle (\lambda I - T)Px, (\lambda I - T)Px \rangle \\ &= |\lambda|^2 \|Px\|^2 - \langle \lambda Px, x \rangle - \langle x, \lambda Px \rangle + \|x\|^2 \\ &= |\lambda|^2 \|Px\|^2 - \lambda \|Px\|^2 - \bar{\lambda} \|Px\|^2 + \|x\|^2 \geq |\lambda - 1|^2 \|Px\|^2. \end{aligned}$$

Thus

$$\|Px\| \leq \frac{1}{|\lambda - 1|} \|(\lambda I - T)Px\|$$

and $\lambda I - T$ has a continuous inverse, i.e., $\rho(T) \supset \mathbb{C} \setminus \{1\}$. Also this estimate shows that

$$\|A_\lambda\| \leq \frac{1}{|\lambda - 1|} \quad \text{for } \lambda \neq 1,$$

from which $A_\lambda \rightarrow 0$ as $\lambda \rightarrow \infty$ and A_λ is analytic at ∞ . Thus $\sigma_e(T) = \{1\}$. (In fact, we can show that $A_\lambda = (\lambda - 1)^{-1}P$ for $\lambda \neq 1$.)

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