EXISTENCE OF MAXIMAL ELEMENT AND EQUILIBRIUM FOR A NONPARACOMPACT N-PERSON GAME

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ABSTRACT. In this paper, we will introduce the concept of L_S -majorized correspondence and prove a new maximal element existence theorem on nonparacompact sets. As applications, we prove a new existence theorem of equilibrium for a nonparacompact 1-person game with L_S -majorized preference correspondences, and then we prove that a nonparacompact N-person game with preference correspondences of class L can be reduced to a 1-person game with L_S -majorized preference correspondences.

1. INTRODUCTION

In the last twenty years, the classical Arrow-Debreu [2] result on the existence of Walrasian equilibria has been generalized in many directions. Bewley [3] proved an equilibrium existence theorem for economies with an infinite dimensional commodity space and Mas-Colell [17] was first to show that the existence of equilibrium can be established without assuming preferences to be total or transitive. Next, by using a maximal element existence theorem, Gale and Mas-Colell [13] gave a proof of the existence of a competitive equilibrium without ordered preferences. Shafer and Sonnenschein [19] gave results in the same direction and they proved 'the Arrow-Debreu lemma for abstract economies' for the case where preferences may not be total or transitive. On the other hand, Borglin and Keiding [4] proved a new existence theorem for a compact abstract economy with KF-majorized preference correspondences, and following their idea, till now there have been a number of generalizations of the existence of equilibria for compact abstract economies.

In 1983, Yannelis and Prabhakar [26] proved the existence of maximal elements over compact subsets of Hausdorff topological vector spaces generalizing the previous results of Fan [8], Sonnenschein [21], Borglin-Keiding [4], and Aliprantis-Brown [1], and they also proved the existence of an equilibrium for an abstract economy defined in Shafer-Sonnenschein [19] and Borglin-Keiding

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[4]. Recently, Ding-Kim-Tan [5, 6] proved some existence theorems of equilibria for noncompact abstract economies with L^* -majorized preferences.

As in [4, 13, 17, 19, 23, 26], in most results on the existence of equilibria for abstract economies, the underlying spaces (commodity spaces or choice sets) are always compact and convex. However, in recent papers [5, 6, 18, 24, 25], the underlying spaces are not always compact, but the paracompactness is needed. Moreover, it should be noted that we will encounter many kinds of preferences in various economic situations; so it is important that we consider several types of preferences and obtain some existence results for such correspondences in noncompact (or nonparacompact) settings.

The purpose of this paper is twofold. First, we will introduce the concept of L_S -majorized correspondence and prove a new maximal element existence theorem on nonparacompact sets. Second, as applications we prove a new existence theorem of equilibrium for a nonparacompact 1-person game with L_S -majorized preference correspondences, and then we prove that a nonparacompact N -person game with preference correspondences of class L can be reduced to a 1-person game with L_S -majorized preference correspondences.

2. PRELIMINARIES

Let A be a subset of a topological space X . We shall denote by 2^A the family of all subsets of A and by $\text{cl } A$ the closure of A in X . If A is a subset of a vector space, we shall denote by $\text{con } A$ the convex hull of A . If A is a nonempty subset of a topological vector space X and $S, T: A \rightarrow 2^X$ are correspondences, then $\text{con } T, \text{cl } T, T \cap S: A \rightarrow 2^X$ are correspondences defined by $(\text{con } T)(x) = \text{con } T(x)$, $(\text{cl } T)(x) = \text{cl } T(x)$, and $(T \cap S)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively.

In [26] Yannelis and Prabhakar introduced the following definition, which generalizes KF correspondences defined in [4, p. 313] and is useful for many applications in mathematical economics (i.e., see [4–6, 23–26]):

Let X be a nonempty subset of a topological vector space. A correspondence $\phi: X \rightarrow 2^X$ is said to be of *class L* [26] if (i) for each $x \in X$, $x \notin \text{con } \phi(x)$, (ii) for each $y \in X$, $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is open in X .

We now introduce the following definition, which generalizes the correspondence of class L .

Definition. Let X be a nonempty subset of a topological vector space and $x \in X$. Let $\phi: X \rightarrow 2^X$ be a given correspondence. A correspondence $\phi_x: X \rightarrow 2^X$ is said to be an L_S -majorant of ϕ at x if there exists an open neighborhood N_x of x in X such that (a) for each $z \in X$, $\phi(z) \subset \phi_x(z)$, (b) for each $z \in N_x$, $z \notin \text{con } \phi_x(z)$, and (c) for each $y \in X$, $\phi_x^{-1}(y)$ is open in X . The correspondence ϕ is said to be L_S -majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an L_S -majorant of ϕ at x .

It is easy to see that every KF correspondence is a correspondence of class L and every correspondence of class L is clearly L_S -majorized. Note that an L_S -majorized correspondence is clearly an L -majorized correspondence defined in [26, p. 240], i.e., our definition has one more extra condition (a). However as in many applications [5, 6, 18, 23, 24] the maximal element existence theorems for L -majorized correspondences always need the compactness (or paracompactness) of the given set X , but we shall not need any paracompact-

ness condition of the set X . And it should be noted that if ϕ is L_s -majorized in X , then $x \notin \text{con } \phi(x)$ for all $x \in X$.

The following simple example shows an L_s -majorized correspondence that is not of class L .

Example. Let $X = [0, 1)$ and $\phi: X \rightarrow 2^X$ be defined by

$$\phi(x) = \begin{cases} \{y \in X : 0 \leq y \leq x^2\}, & x \in (0, 1), \\ \emptyset, & x = 0. \end{cases}$$

Then ϕ is not of class L since $\phi^{-1}(y)$ is not open in X for any $y \in (0, 1)$. For any $x \in (0, 1)$, let $N_x = X$ be an open neighborhood of x in X , and define $\phi_x: X \rightarrow 2^X$ by

$$\phi_x(z) = \begin{cases} \{y \in X : 0 \leq y < x\}, & z \in (0, 1), \\ \emptyset, & z = 0. \end{cases}$$

Then it is easy to see that ϕ_x is an L_s -majorant of ϕ at x , and hence ϕ is a L_s -majorized correspondence.

Now we recall the following general definitions of equilibrium theory in mathematical economics. Let I be a finite or an infinite set of agents. For each $i \in I$, let X_i be a nonempty set of actions. A *generalized game* (or an *abstract economy*) $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ is defined as a family of ordered quadruple (X_i, A_i, B_i, P_i) where X_i is a nonempty topological vector space (a choice set), $A_i, B_i: \prod_{j \in I} X_j \rightarrow 2^{X_i}$ are constraint correspondences and $P_i: \prod_{j \in I} X_j \rightarrow 2^{X_i}$ is a preference correspondence. An *equilibrium* for Γ is a point $\hat{x} \in X = \prod_{i \in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in \text{cl } B_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$. When $A_i = B_i$ for each $i \in I$, our definitions of an abstract economy and an equilibrium coincide with the standard definitions, e.g., in [4, 19, 23, 24, 26]. For each $i \in I$, $P'_i: X \rightarrow 2^{X_i}$ will denote the correspondence defined by $P'_i(x) = \{y \in X : y_i \in P_i(x)\} (= \pi_i^{-1}(P_i(x)))$, where $\pi_i: X \rightarrow X_i$ is the i th projection). In particular, when $I = \{1, \dots, n\}$, we may call Γ an N -person game.

3. EXISTENCE OF A MAXIMAL ELEMENT

We shall first need the following lemma, which is a generalization of earlier Fan's maximal element existence theorem appeared in [4, p. 313] and also represents a maximal element version of his recent intersection result [9, Theorem 4]. For completeness, we give a proof.

Lemma. Let X be a nonempty convex subset of a Hausdorff topological vector space and $T: X \rightarrow 2^X$ be a correspondence such that for each $x \in X$, $x \notin \text{con } T(x)$ and for each $y \in X$, $T^{-1}(y)$ is open in X . Suppose further that there exists a nonempty compact subset D of X and a nonempty compact convex subset X_0 of X such that $X_0 \cap \text{con } T(x) \neq \emptyset$ for all $x \in X \setminus D$.

Then there exists a point $\hat{x} \in X$ such that $T(\hat{x}) = \emptyset$.

Proof. Suppose that the conclusion is false; then for each $x \in X$ we would have $T(x) \neq \emptyset$ so that $\text{con } T(x) \neq \emptyset$. By Lemma 5.1 in [26], $(\text{con } T)^{-1}(y)$ is open in X for each $y \in X$. Now we define a correspondence

$$F(x) := X \setminus (\text{con } T)^{-1}(x) \quad \text{for each } x \in X.$$

Then for each $x \in X$, $F(x)$ is closed in X . Now we claim that $\bigcap_{x \in X_0} F(x) \subset D$. Suppose the contrary, i.e., there exists a point $y \in \bigcap_{x \in X_0} F(x)$ such that $y \in X \setminus D$. Then we have

$$\begin{aligned} y \in \bigcap_{x \in X_0} F(x) &\Leftrightarrow y \in \bigcap_{x \in X_0} (X \setminus (\text{con } T)^{-1}(x)) \\ &\Leftrightarrow y \in X \setminus \bigcup_{x \in X_0} (\text{con } T)^{-1}(x) \\ &\Leftrightarrow X_0 \cap (\text{con } T)(y) = \emptyset, \end{aligned}$$

which contradicts the assumption. Therefore $\bigcap_{x \in X_0} F(x)$ is a closed subset of a compact set D so that $\bigcap_{x \in X_0} F(x)$ is compact. Since $\text{con } T(x) \neq \emptyset$ for each $x \in X$, we have $\bigcup_{x \in X} (\text{con } T)^{-1}(x) = X$, so that

$$\bigcap_{x \in X} F(x) = \bigcap_{x \in X} (X \setminus (\text{con } T)^{-1}(x)) = X \setminus \bigcup_{x \in X} (\text{con } T)^{-1}(x) = \emptyset.$$

Therefore, by Theorem 4 in [9], F cannot be a KKM-map, so that there exists a convex combination $\omega = \sum_{i=1}^n \lambda_i x_i \notin \bigcup_{i=1}^n F(x_i)$. Hence we have $\omega \in X \setminus \bigcup_{i=1}^n F(x_i) = \bigcap_{i=1}^n (\text{con } T)^{-1}(x_i)$, so that $\omega \in (\text{con } T)^{-1}(x_i)$ for each $i = 1, \dots, n$. Therefore $x_i \in \text{con } T(\omega)$ for each $i = 1, \dots, n$. Since $\text{con } T(\omega)$ is convex, we have $\omega = \sum_{i=1}^n \lambda_i x_i \in \text{con } T(\omega)$, which contradicts the assumption. Q.E.D.

The existence of maximal element is essential in proving existence of equilibria in generalized games as in [4–6, 18, 23–26]. Recently, using Fan's intersection theorem [9] or the fixed point theorem as in [26], several authors further generalize maximal element existence results in the paracompact settings (i.e., see [5, 6, 18, 23, 24]).

By modifying the method in [5], we shall prove the following new maximal element existence theorem for L_s -majorized correspondences without assuming the paracompactness of the given set X , which is essential for the existence of equilibria for a nonparacompact 1-person game.

Theorem 1. *Let X be a nonempty convex subset of a Hausdorff topological vector space, D be a nonempty compact subset of X , and $y_0 \in D$. Let $\phi: X \rightarrow 2^X$ be an L_s -majorized correspondence such that for each $x \in X \setminus D$, $y_0 \in \text{con } \phi(x)$.*

Then there exists a maximal element $\hat{x} \in X$, i.e., $\phi(\hat{x}) = \emptyset$.

Proof. Suppose that for each $x \in X$, $\phi(x) \neq \emptyset$. Since ϕ is L_s -majorized, for each $x \in D$, there exists an L_s -majorant ϕ_x and an open neighborhood N_x of x in X such that

- (i) for each $z \in X$, $\phi(z) \subset \phi_x(z)$,
- (ii) for each $z \in N_x$, $z \notin \text{con } \phi_x(z)$,
- (iii) for each $y \in X$, $\phi_x^{-1}(y)$ is open in X .

The family $\{N_x : x \in D\}$ is an open covering of D , which by the compactness of D , contains a finite subcover $\{N_{x_i} : i \in I\}$, where I is a finite set. Then $U = \bigcup_{i \in I} N_{x_i}$ is an open set in X containing D . For each $i \in I$, denote by $\text{con } \phi'_{x_i}$ the restriction of $\text{con } \phi_{x_i}$ on U . Then for each $i \in I$ and for each $y \in X$,

$$\begin{aligned} (\text{con } \phi'_{x_i})^{-1}(y) &= \{z \in U : y \in \text{con } \phi'_{x_i}(z)\} \\ &= \{z \in U : y \in \text{con } \phi_{x_i}(z)\} = U \cap (\text{con } \phi_{x_i})^{-1}(y) \end{aligned}$$

is open in X by Lemma 5.1 in [26] and condition (iii). Now we define $\Phi: X \rightarrow 2^X$ by

$$\Phi(z) = \begin{cases} \bigcap_{i \in I} \text{con } \phi'_{x_i}(z) & \text{if } z \in U, \\ \{y_0\} & \text{if } z \notin U. \end{cases}$$

Then for each $z \in X$, $z \notin \text{con } \Phi(z)$ by the following reason: in case $z \in U$, then $z \in N_{x_i}$ for some $i \in I$. Therefore $z \notin \text{con } \phi'_{x_i}(z) = \text{con } \phi_{x_i}(z)$ so that $z \notin \text{con } \Phi(z)$. On the other hand, in case $z \in X \setminus U$, then $z \in X \setminus D$. Therefore we have $z \neq y_0$, so that $z \notin \text{con } \Phi(z)$.

Now we show that for each $y \in X$, $\Phi^{-1}(y)$ is open in X . Indeed, in case $y \neq y_0$, we have

$$\begin{aligned} \Phi^{-1}(y) &= \{z \in X : y \in \Phi(z)\} \\ &= \{z \in U : y \in \Phi(z)\} \cup \{z \in X \setminus U : y \in \Phi(z)\} \\ &= \{z \in U : y \in \Phi(z)\} = \left\{ z \in U : y \in \bigcap_{i \in I} \text{con } \phi'_{x_i}(z) \right\} \\ &= U \cap \left[\bigcap_{i \in I} (\text{con } \phi'_{x_i})^{-1}(y) \right] \end{aligned}$$

is open in X since $(\text{con } \phi'_{x_i})^{-1}(y)$ is open in X for each $i \in I$.

In case $y = y_0$, we have

$$\begin{aligned} \Phi^{-1}(y_0) &= \{z \in X : y_0 \in \Phi(z)\} \\ &= \{z \in U : y_0 \in \Phi(z)\} \cup \{z \in X \setminus U : y_0 \in \Phi(z)\} \\ &= \left[U \cap \left(\bigcap_{i \in I} (\text{con } \phi'_{x_i})^{-1}(y_0) \right) \right] \cup (X \setminus U) \\ &= \left[U \cap \left(\bigcap_{i \in I} (\text{con } \phi_{x_i})^{-1}(y_0) \right) \right] \cup (X \setminus U) \\ &= \left[\bigcap_{i \in I} (\text{con } \phi_{x_i})^{-1}(y_0) \right] \cup (X \setminus U). \end{aligned}$$

By the assumption and condition (i), for each $z \in X \setminus D$,

$$y_0 \in \text{con } \phi(z) \subset \text{con } \phi_{x_i}(z) \quad \text{for each } i \in I.$$

Therefore for each $i \in I$, $X \setminus D \subset (\text{con } \phi_{x_i})^{-1}(y_0)$, so that $X \setminus U \subset (\text{con } \phi_{x_i})^{-1}(y_0)$. Therefore,

$$\begin{aligned} \Phi^{-1}(y_0) &= \left[\bigcap_{i \in I} (\text{con } \phi_{x_i})^{-1}(y_0) \right] \cup (X \setminus U) \\ &= \bigcap_{i \in I} (\text{con } \phi_{x_i})^{-1}(y_0) \end{aligned}$$

is open in X since $(\text{con } \phi_{x_i})^{-1}(y_0)$ is open in X for each $i \in I$. Therefore $\Phi: X \rightarrow 2^X$ is a correspondence of class L .

Finally we shall show that for each $z \in X \setminus D$, $y_0 \in \text{con } \Phi(z)$. First, if $z \in X \setminus U$, then $y_0 \in \text{con } \Phi(z) = \{y_0\}$. For $z \in U \setminus D$, we must show that $y_0 \in \text{con } \Phi(z)$. Indeed, for every $i \in I$,

$$\phi(z) \subset \phi'_{x_i}(z) = \phi_{x_i}(z) \quad \text{for each } z \in U,$$

so that $\phi(z) \subset \Phi(z)$ for each $z \in U$. By the assumption, $y_0 \in \text{con } \phi(z) \subset \text{con } \Phi(z)$ for each $z \in U \setminus D$. Therefore, for all $z \in X \setminus D$, $y_0 \in \text{con } \Phi(z)$. Therefore by letting $X_0 = \{y_0\}$, all the hypotheses of Lemma are satisfied, so that there exists a point $\hat{x} \in X$ such that $\Phi(\hat{x}) = \emptyset$. This implies that $\hat{x} \in U$. Since $\phi(z) \subset \Phi(z)$ for each $z \in U$, $\phi(\hat{x}) = \emptyset$, which is a contradiction. Hence there must exist a maximal element $\hat{x} \in X$ such that $\phi(\hat{x}) = \emptyset$. Q.E.D.

Remark. As we have seen in the previous example, the correspondence $\phi : X \rightarrow 2^X$ defined by

$$\phi(x) = \begin{cases} \{y \in X : 0 \leq y \leq x^2\}, & x \in (0, 1), \\ \emptyset, & x = 0, \end{cases}$$

is L_s -majorized but not of class L . Therefore any maximal element existence theorem for class L (i.e., Lemma of Theorem 1 or results in [13, 23, 24, 26]) cannot be applicable. However, by letting $D = \{y_0\} = \{0\}$ in Theorem 1, we have a maximal element $0 \in [0, 1)$ such that $\phi(0) = \emptyset$.

In case X is compact convex, we have the following immediate consequence which generalizes Fan's maximal element existence theorem appeared in [4]:

Corollary. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and let $\phi : X \rightarrow 2^X$ be a correspondence of class L ; i.e., for each $x \in X$, $x \notin \text{con } \phi(x)$ and for each $y \in X$, $\phi^{-1}(y)$ is open in X . Then there exists a point $\hat{x} \in X$ such that $\phi(\hat{x}) = \emptyset$.*

4. EXISTENCE OF EQUILIBRIUM FOR A NONPARACOMPACT N-PERSON GAME

As we have seen in [4, 5, 6, 18, 23, 24, 26], maximal element existence theorems are essential tools in the proofs of the existence of equilibria for generalized games.

As an application of Theorem 1, we obtain the following existence theorem of equilibrium for a nonparacompact 1-person game with an L_s -majorized preference correspondence.

Theorem 2. *Let X be a nonempty convex subset of a Hausdorff topological vector space (a choice set), D be a nonempty compact subset of X , and $y_0 \in D$. Let $A, B : X \rightarrow 2^X$ be constraint correspondences and $P : X \rightarrow 2^X$ be a preference correspondence satisfying*

- (1) $A \cap P$ is L_s -majorized;
- (2) for each $x \in X$, $A(x)$ is nonempty and $\text{con } A(x) \subset B(x)$;
- (3) for each $y \in X$, $A^{-1}(y)$ is open in X ;
- (4) the correspondence $\text{cl } B : X \rightarrow 2^X$ is upper semicontinuous;
- (5) for each $x \in X \setminus D$, $y_0 \in (A \cap P)(x)$.

Then there exists an equilibrium choice $\hat{x} \in X$, i.e., $\hat{x} \in \text{cl } B(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Proof. Let $F = \{x \in X : x \in \text{cl } B(x)\}$; then F is closed in X since $\text{cl } B$ is upper semicontinuous. Define $\psi: X \rightarrow 2^X$ by

$$\psi(x) = \begin{cases} \text{con}(A \cap P)(x) & \text{if } x \in F, \\ \text{con } A(x) & \text{if } x \notin F. \end{cases}$$

If $x \notin F$, then $X \setminus F$ is an open neighborhood of x in X such that for each $z \in X \setminus F$, $z \notin \text{cl } B(z)$ and $\psi(z) = \text{con } A(z)$; since $\text{con } A(z) \subset B(z)$, we have $z \notin \text{con } A(z)$. And it is clear that $\psi(z) \subset \text{con } A(z)$. Since $A^{-1}(y)$ is open in X , $(\text{con } A)^{-1}(y)$ is also open in X for all $y \in X$ by Lemma 5.1 in [26], so that $\text{con } A$ is an L_S -majorant of ψ at x .

Now suppose that $x \in F$ and $\psi(x) = \text{con}(A \cap P)(x) \neq \emptyset$; then by the assumption (1), there exist $\phi_x: X \rightarrow 2^X$ and an open neighborhood N_x of x in X such that

- (i) $(A \cap P)(z) \subset \phi_x(z)$ for all $z \in X$,
- (ii) $z \notin \text{con } \phi_x(z)$ for all $z \in N_x$,
- (iii) $\phi_x^{-1}(y)$ is open in X for all $y \in X$.

Define $\phi'_x: X \rightarrow 2^X$ by

$$\phi'_x(z) = \begin{cases} \text{con } \phi_x(z) \cap \text{con } A(z), & \text{if } z \in F, \\ \text{con } A(z), & \text{if } z \notin F. \end{cases}$$

Then we have the following facts: (a) for each $z \in N_x$, it is easy to see that $\psi(z) \subset \phi'_x(z)$, (b) for each $z \in N_x$, $z \notin \text{con } \phi'_x(z)$, and (c) for each $y \in X$,

$$\begin{aligned} (\phi'_x)^{-1}(y) &= \{z \in X : y \in \phi'_x(z)\} \\ &= \{z \in F : y \in \phi'_x(z)\} \cup \{z \in X \setminus F : y \in \phi'_x(z)\} \\ &= \{z \in F : y \in [\text{con } \phi_x(z) \cap \text{con } A(z)]\} \cup \{z \in X \setminus F : y \in \text{con } A(z)\} \\ &= [F \cap (\text{con } A)^{-1}(y) \cap (\text{con } \phi_x)^{-1}(y)] \cup [(X \setminus F) \cap (\text{con } A)^{-1}(y)] \\ &= [(\text{con } \phi_x)^{-1}(y) \cup (X \setminus F)] \cap (\text{con } A)^{-1}(y) \end{aligned}$$

is open in X .

This shows that ϕ'_x is an L_S -majorant of ψ at x . Therefore ψ is L_S -majorized. By Theorem 1, there exists a point $\hat{x} \in X$ such that $\psi(\hat{x}) = \emptyset$. Since $A(x) \neq \emptyset$ for each $x \in X$, we must have $\hat{x} \in F$, i.e., $\hat{x} \in \text{cl } B(\hat{x})$ and $\psi(\hat{x}) = \text{con}(A \cap P)(\hat{x}) = \emptyset$, so that $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. Q.E.D.

In case X is compact convex and $A = B$, by letting $D = X$ we have the following immediate consequence of Theorem 2:

Corollary 1. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space and let $P, A: X \rightarrow 2^X$ be two correspondences satisfying the following conditions:*

- (1) $A \cap P$ is L_S -majorized;
- (2) for each $x \in X$, $A(x)$ is nonempty convex;
- (3) for each $y \in X$, $A^{-1}(y)$ is open in X ;
- (4) the correspondence $\text{cl } A: X \rightarrow 2^X$ is upper semicontinuous.

Then there exists an equilibrium choice $\hat{x} \in X$, i.e., $\hat{x} \in \text{cl } A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Let I be a finite set of agents and X_i be a Hausdorff topological vector space. Let $X = \prod_{i \in I} X_i$. For a given correspondence $A_i: X \rightarrow 2^{X_i}$, recall

that a correspondence $A'_i: X \rightarrow 2^X$ is defined by $A'_i(x) = \{y \in X : y_i \in A_i(x)\}$ ($= \pi_i^{-1}(A_i(x))$ where $\pi_i: X \rightarrow X_i$ is the i th projection). Then it is easy to show that the following two conditions are equivalent:

- (1) A'_i is a correspondence of class L ;
- (2) for each $x \in X$, $x_i \notin \text{con } A_i(x)$ and for each $y \in X_i$, $A_i^{-1}(y)$ is open in X .

We shall now show that the case of n agents ($n > 1$) in a nonparacompact abstract economy with preference correspondences of class L can be reduced to a 1-person game with L_s -majorized preference correspondence (i.e., Theorem 2).

Theorem 3. Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy where I is a finite set such that for each $i \in I$,

- (1) X_i is a nonempty convex subset of a Hausdorff topological vector space and D_i is a nonempty compact subset of X_i ;
- (2) for each $x \in X = \prod_{i \in I} X_i$, $A_i(x)$ is nonempty and $\text{con } A_i(x) \subset B_i(x)$;
- (3) for each $y \in X_i$, $A_i^{-1}(y)$ is open in X ;
- (4) the correspondence $\text{cl } B_i: X \rightarrow 2^{X_i}$ is upper semicontinuous;
- (5) the correspondence $A'_i \cap P'_i$ is of class L ;
- (6) there exists a point $y_0 \in \prod_{i \in I} D_i$ such that for each $x \in X \setminus \prod_{i \in I} D_i$, $y_0 \in (A'_i \cap P'_i)(x)$.

Then Γ has an equilibrium choice $\hat{x} \in X$, i.e., for each $i \in I$, $\hat{x}_i \in \text{cl } B_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Proof. Let $D = \prod_{i \in I} D_i$. Then D is compact by Tychonoff's product theorem. Define the correspondences $A, B, P: X \rightarrow 2^X$ by

$$A(x) = \prod_{i \in I} A_i(x), \quad B(x) = \prod_{i \in I} B_i(x),$$

and

$$P(x) = \begin{cases} \bigcap_{i \in I(x)} P'_i(x), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{if } I(x) = \emptyset, \end{cases}$$

where $I(x) = \{i \in I : A_i(x) \cap P_i(x) \neq \emptyset\}$.

By (2), for each $x \in X$, $A(x)$ is nonempty and $\text{con } A(x) \subset B(x)$. By (3), for each $y \in X$, $A^{-1}(y) = \bigcap_{i \in I} A_i^{-1}(y)$ is open in X . Moreover, since $\text{cl } B(x) = \text{cl } \prod_{i \in I} B_i(x) = \prod_{i \in I} \text{cl } B_i(x)$ for each $x \in X$, by (4) and Lemma 3 in [7], $\text{cl } B: X \rightarrow 2^X$ is also upper semicontinuous. By (6), for each $x \in X \setminus D$, $I(x) = I$ and $y_0 \in (A \cap P)(x)$. Therefore the hypotheses (2), (3), (4), and (5) of Theorem 2 are satisfied. Finally we shall show that the correspondence $A \cap P$ is L_s -majorized. Let $x \in X$ with $(A \cap P)(x) \neq \emptyset$, i.e., $\prod_{i \in I(x)} (A_i \cap P_i)(x) \neq \emptyset$ and $I(x) \neq \emptyset$. We first show that there exists an open neighborhood N_x of x in X such that $I(x) \subset I(z)$ (and hence $I(z) \neq \emptyset$) for all $z \in N_x$. Indeed, let $i \in I(x)$; as $A'_i(x) \cap P'_i(x) \neq \emptyset$, take any $y \in A'_i(x) \cap P'_i(x)$, then $x \in (A'_i \cap P'_i)^{-1}(y)$. Let $N_i = (A'_i \cap P'_i)^{-1}(y)$, then by assumption (5), N_i is an open neighborhood of x . Let $N_x = \bigcap_{i \in I(x)} N_i$, then N_x is an open neighborhood of x in X . If $z \in N_x$, then for each $i \in I(x)$, $z \in N_i = (A'_i \cap P'_i)^{-1}(y)$ so that

$y \in (A'_i \cap P'_i)(z)$ and hence $(A'_i \cap P'_i)(z) \neq \emptyset$; that is, $i \in I(z)$. This shows that $I(x) \subset I(z)$ for all $z \in N_x$. Now fix any $i_0 \in I(x)$. Then for any $z \in N_x$, we have

$$\begin{aligned} (A \cap P)(z) &\subset \bigcap_{i \in I(z)} A'_i(z) \cap P'_i(z) \\ &\subset \bigcap_{i \in I(x)} A'_i(z) \cap P'_i(z) \quad (\text{since } I(x) \subset I(z)) \\ &\subset (A'_{i_0} \cap P'_{i_0})(z). \end{aligned}$$

Since X is regular and N_x is an open neighborhood of x in X , there exists an open neighborhood M_x of x in X such that $M_x \subset \text{cl } M_x \subset N_x$.

We now define a correspondence $\phi_x: X \rightarrow 2^X$ by

$$\phi_x(z) = \begin{cases} (A'_{i_0} \cap P'_{i_0})(z), & \text{if } z \in \text{cl } M_x, \\ X, & \text{if } z \notin \text{cl } M_x. \end{cases}$$

Then it is easy to show that $(A \cap P)(z) \subset \phi_x(z)$ for each $z \in X$. And, for each $z \in M_x$, $z \notin \text{con } \phi_x(z)$ by assumption (5). For each $y \in X$,

$$\begin{aligned} \phi_x^{-1}(y) &= \{z \in X : y \in \phi_x(z)\} \\ &= \{z \in \text{cl } M_x : y \in \phi_x(z)\} \cup \{z \in X \setminus \text{cl } M_x : y \in \phi_x(z)\} \\ &= \{z \in \text{cl } M_x : y \in (A'_{i_0} \cap P'_{i_0})(z)\} \cup (X \setminus \text{cl } M_x) \\ &= [\text{cl } M_x \cap (A'_{i_0} \cap P'_{i_0})^{-1}(y)] \cup (X \setminus \text{cl } M_x) \\ &= (A'_{i_0} \cap P'_{i_0})^{-1}(y) \cup (X \setminus \text{cl } M_x) \end{aligned}$$

is open in X by (5). Therefore ϕ_x is an L_s -majorant of $A \cap P$ at x . (Note that M_x is a desired open neighborhood of x in X .)

Therefore $A \cap P$ is L_s -majorized, so that all the hypotheses of Theorem 2 are satisfied. By Theorem 2, there exists an equilibrium point $\hat{x} \in X$ such that $\hat{x}_i \in \text{cl } B(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. Hence for each $i \in I$, $\hat{x}_i \in \text{cl } B_i(\hat{x})$. Next we shall show that $I(\hat{x}) = \emptyset$. Suppose $I(\hat{x}) \neq \emptyset$. Since $A(\hat{x}) \cap P(\hat{x}) = \emptyset$ implies $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for at least one $i \in I(\hat{x})$, which contradicts the definition of $I(\hat{x})$. Therefore we must have $I(\hat{x}) = \emptyset$, i.e., $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for each $i \in I$. Q.E.D.

Remarks. (1) Since X_i is neither compact nor paracompact, our Theorem 3 is comparable with the previous equilibrium existence results in [4, 5, 6, 18, 23, 24, 26] and as remarked before, equilibrium existence theorems for compact (or paracompact) abstract economies cannot be applied to our settings.

(2) We do not know whether assumption (5) can be relaxed to L_s -majorized correspondences and we also do not know whether the set of agents I may be infinite.

In case X_i is compact convex and $A_i = B_i$ for each $i \in I$, by letting $D_i = X_i$ for each $i \in I$, we have the following immediate consequence of Theorem 3:

Corollary 2. Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy where I is a finite set such that for each $i \in I$,

- (1) X_i is a nonempty compact convex subset of a Hausdorff topological vector space;
- (2) for each $x \in X = \prod_{i \in I} X_i$, $A_i(x)$ is a nonempty convex subset of X_i ;

- (3) for each $y \in X_i$, $A_i^{-1}(y)$ is open in X ;
- (4) the correspondence $\text{cl } A_i: X \rightarrow 2^{X_i}$ is upper semicontinuous;
- (5) the correspondence $A'_i \cap P'_i$ is of class L .

Then Γ has an equilibrium choice $\hat{x} \in X$, i.e., for each $i \in I$, $\hat{x}_i \in \text{cl } A_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

When the set of agents I is finite, Corollary 2 answers the question raised in [26, p. 243] positively; i.e., Corollary 2 holds true in nonmetrizable and nonlocally convex spaces.

REFERENCES

1. C. Aliprantis and D. Brown, *Equilibria in markets with a Riesz space of commodities*, J. Math. Econom. **11** (1983), 189–207.
2. K. J. Arrow and G. Debreu, *Existence of an equilibrium for a competitive economy*, Econometrica **22** (1954), 265–290.
3. T. F. Bewley, *Existence of equilibria in economies with infinitely many commodities*, J. Econom. Theory **4** (1972), 514–540.
4. A. Borglin and H. Keiding, *Existence of equilibrium actions and of equilibrium: A note on the 'new' existence theorem*, J. Math. Econom. **3** (1976), 313–316.
5. X. P. Ding, W. K. Kim, and K.-K. Tan, *Equilibria of non-compact generalized games with L^* -majorized preferences*, J. Math. Anal. Appl., in press.
6. —, *Equilibrium theorems on non-compact abstract economies with L^* -majorized preference correspondences*, in preparation.
7. K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 131–136.
8. —, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310.
9. —, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519–537.
10. P. Fishburn and R. Rosenthal, *Noncooperative games and nontransitive preferences*, Math. Social Sci. **12** (1986), 1–7.
11. M. Florenzano, *On the existence of equilibria in economies with an infinite dimensional commodity spaces*, J. Math. Econom. **12** (1983), 207–219.
12. —, *On an extension of the Gale-Nikaido-Debreu lemma*, Econom. Lett. **25** (1987), 51–53.
13. D. Gale and A. Mas-Colell, *An equilibrium existence for a general model without ordered preferences*, J. Math. Econom. **2** (1975), 9–15.
14. A. Kajii, *Note on equilibrium without ordered preferences in topological vector space*, Econom. Lett. **27** (1988), 1–4.
15. H. Keiding, *Existence of economic equilibria: New results and open problems*, Lecture Notes in Econom. and Math. System, vol. 226, Springer-Verlag, Berlin and New York, 1984, pp. 223–243.
16. M. Ali Khan, *On extensions of the Cournot-Nash theorem*, Advances in Equilibrium Theory (C. Aliprantis et al., eds.), Springer-Verlag, Berlin, 1985.
17. A. Mas-Colell, *An equilibrium existence theorem without complete or transitive preferences*, J. Math. Econom. **1** (1974), 237–246.
18. G. Mehta, *Fixed points, equilibria and maximal elements in linear topological spaces*, Comm. Math. Univ. Carolinae **28** (1987), 377–385.
19. W. Shafer and H. Sonnenschein, *Equilibrium in abstract economies without ordered preferences*, J. Math. Econom. **2** (1975), 345–348.
20. W. Shafer, *Equilibrium in economies without ordered preferences or free disposal*, J. Math. Econom. **3** (1976), 135–137.
21. H. Sonnenschein, *Demand theory without transitive preference with applications to the theory of competitive equilibrium*, Preferences, Utility and Demand (J. Chipman et al., eds.), Harcourt Brace Jovanovich, New York, 1971.

22. E. Tarafdar and G. Mehta, *A generalized version of the Gale-Nikaido-Debreu theorem*, Comm. Math. Univ. Carolinae **28** (1987), 655–659.
23. S. Toussaint, *On the existence of equilibria in economies with infinitely many commodities and without ordered preferences*, J. Econom. Theory **33** (1984), 98–115.
24. C. I. Tulcea, *On the equilibriums of generalized games*, The Center for Mathematical Studies in Economics and Management Science, Paper No. 696, 1986.
25. —, *On the approximation of upper semicontinuous correspondences and the equilibrium of the generalized games*, J. Math. Anal. Appl. **136** (1988), 267–289.
26. N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Econom. **12** (1983), 233–245.
27. —, *Erratum*, J. Math. Econom. **13** (1984), 305.
28. N. C. Yannelis, *Maximal elements over non-compact subsets of linear topological spaces*, Econom. Lett. **17** (1985), 133–136.
29. —, *On a market equilibrium theorem with an infinite number of commodities*, J. Math. Anal. Appl. **108** (1985), 595–599.

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