A FULL CHARACTERIZATION OF STABLE UNIT BALLS IN ORLICZ SPACES

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ABSTRACT. We give a full characterization of stable unit balls in Orlicz spaces, with Luxemburg norm, completing the partial results given by the author in *Stable unit balls in Orlicz spaces*, Proc. Amer. Math. Soc. **109** (1990), 97–104 and by M. Wisla in *Extreme points and stable unit balls in Orlicz sequence spaces*, Arch. Math. **56** (1991), 482–490.

If X is a Banach space, denote by B_X and S_X the closed unit ball and unit sphere of X , respectively. If A is a set, χ_A will be its characteristic function and $f|_A$ the restriction of a function f to A . We will consider a positive σ -finite measure space (Ω, Σ, μ) and an Orlicz function $\varphi: \mathbb{R} \rightarrow [0, +\infty]$ convex, even, left-continuous for $x > 0$, $\varphi(0) = 0$, and $\varphi(x) > 0$ for some $x \in \mathbb{R}$. Let $a_0 = \sup\{x \geq 0: \varphi(x) = 0\}$ and $\tau = \sup\{x \geq 0: \varphi(x) < \infty\}$. The Orlicz space $L^\varphi(\mu)$ is the set of (equivalence classes of) measurable functions $f: \Omega \rightarrow \mathbb{R}$ such that the functional $I_\varphi(\lambda f) = \int_\Omega \varphi(\lambda f) d\mu < +\infty$, for some $\lambda > 0$, equipped with the Luxemburg norm $\|f\|_\varphi = \inf\{\lambda > 0: I_\varphi(f/\lambda) \leq 1\}$. Suppose that $\tau > 0$ (if $\tau = 0$ then $L^\varphi(\mu) = \{0\}$) and put $B_\varphi = B_{L^\varphi(\mu)}$, $S_\varphi = S_{L^\varphi(\mu)}$, and $E^\varphi(\mu) = \{f \in L^\varphi(\mu): I_\varphi(\lambda f) < \infty, \forall \lambda > 0\}$.

A convex subset C of a topological vector space is called *stable* if the mid-point map $\Phi_{1/2}: C \times C \rightarrow C$ is open, where $\Phi_{1/2}(x, y) = \frac{1}{2}(x + y)$. Stable sets were studied in [1, 4]. The stability of the unit ball B_φ of an Orlicz space was considered first in [2], where it was obtained that B_φ is stable if $I_\varphi(f) = 1$ for each $f \in S_\varphi$. In [5] it was proved that the unit ball B_φ of the Orlicz sequence space L^φ is stable iff l^φ is isometric to l^∞ or φ satisfies the Δ_2 condition at 0. In this paper we obtain a full characterization of the stability of the unit ball B_φ , connecting the following facts: (a) B_φ is stable; (b) $E^\varphi(\mu) = L^\varphi(\mu)$; and (c) $I_\varphi(f) = 1$ for each $f \in S_\varphi$. The connection between (b) and the condition Δ_2 or δ_2 , depending on the measure μ , can be found in [3, Theorems 8.13 and 8.14].

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For $L^\varphi(\mu)$ consider the following statement:

- (*) There exist $x, y, z \in S_\varphi$ with $z = \frac{1}{2}(x+y)$, $\varepsilon > 0$, and $B \in \Sigma$ such that (i) $\int_B \varphi(u) d\mu > 0$ for all $u \in B_\varphi$ with $\|u - x\|_\varphi < \varepsilon$; (ii) for every $\delta > 0$ there exists $w \in B_\varphi$ with $\|w - z\|_\varphi \leq \delta$ and $\int_{\Omega \setminus B} \varphi(w) d\mu = 1$.

Lemma 1. *If $L^\varphi(\mu)$ satisfies (*) then B_φ is not stable.*

Proof. Fix $\delta > 0$ and $w \in B_\varphi$ such that $\|w - z\|_\varphi \leq \delta$ and $\int_{\Omega \setminus B} \varphi(w) d\mu = 1$. Take $u, v \in B_\varphi$ with $\|x - u\|_\varphi < \varepsilon$ and $\|y - v\|_\varphi < \varepsilon$. We claim that $\frac{1}{2}(u+v) \neq w$. Indeed, if $\frac{1}{2}(u+v) = w$, applying (*) and the convexity of φ , we obtain

$$\begin{aligned} 2 &\geq I_\varphi(u) + I_\varphi(v) = \int_B [\varphi(u) + \varphi(v)] d\mu + 2 \int_{\Omega \setminus B} \frac{1}{2} [\varphi(u) + \varphi(v)] d\mu \\ &\geq \int_B [\varphi(u) + \varphi(v)] d\mu + 2 \int_{\Omega \setminus B} \varphi\left(\frac{1}{2}(u+v)\right) d\mu > 2, \end{aligned}$$

a contradiction. Hence $\{w \in B_\varphi : \|z - w\|_\varphi \leq \delta\} \not\subseteq \{\frac{1}{2}(u+v) : u, v \in B_\varphi, \|x - u\|_\varphi < \varepsilon, \|y - v\|_\varphi < \varepsilon\}$. As δ is arbitrary, we conclude that B_φ is not stable. $\square$

If $\tau < \infty$ and $\varphi(\tau) < \infty$, we say that $f \in S_\varphi$ is *nonterminal* if there exists $0 < \beta < \tau$ such that $|f(t)| \leq \tau - \beta$ a.e. $t \in \Omega$. Otherwise, if

$$\mu(\{t \in \Omega : |f(t)| \geq \tau - 1/n\}) > 0, \quad \forall n \geq 1,$$

f is said to be *terminal*.

Lemma 2. *The statement (*) holds if there exists $f \in S_\varphi$ such that $I_\varphi(f) < 1$ and one of the following conditions is fulfilled:*

- (1) $\lim_{t \rightarrow \tau^-} \varphi(t) = +\infty$;
- (2) $\mu(\Omega) = +\infty$, $\varphi(\tau) < +\infty$, and f is nonterminal.

Proof. (A) Assume that

(A1) There exist $g \in S_\varphi$ with $I_\varphi(g) < 1$ and a sequence of pairwise disjoint subsets $\{B_n\}_{n \geq 1} \subseteq \Sigma$, with $0 < \mu(B_1) < +\infty$ and $\sum_{n \geq 2} \int_{B_n} \varphi(\lambda g) d\mu = +\infty$, $\forall \lambda > 1$, and a number $\gamma > 0$ such that $\int_{B_1} \varphi(a_0 + \gamma) d\mu + \sum_{n \geq 2} \int_{B_n} \varphi(g) d\mu \leq 1$.

(A2) For each $\delta > 0$ we can choose an integer $q \geq 2$ and a sequence of measurable maps $c_m : \{0\} \cup \mathbb{R}^+ \rightarrow \mathbb{R}$, $c_m(t) \geq 0$ if $t \leq \tau$, such that $\delta^{-1} \cdot c_m(|g(t)|) < 1$, $t \in B_i$, $i \geq q$, and $F(m) < +\infty$ but $F(m) \uparrow +\infty$ whereas $m \rightarrow +\infty$, $m \in \mathbb{N}$, where

$$F(m) = \sum_{n=2}^{q-1} \int_{B_n} \varphi(g) d\mu + \sum_{n=q}^m \int_{B_n} \varphi([1 + c_m(|g|)]g) d\mu + \sum_{n>m} \int_{B_n} \varphi(g) d\mu.$$

We claim that (A1) and (A2) imply (*). Indeed, let $B_0 = \Omega \setminus \bigcup_{n \geq 1} B_n$ and define $x, y, z \in L^\varphi(\mu)$

$$\begin{aligned} z|_{B_1} &= a_0; & x|_{B_1} &= a_0 + \gamma; & y|_{B_1} &= a_0 - \gamma; \\ z|_{B_n} &= x|_{B_n} = y|_{B_n} = g, & n &\geq 2; \\ z|_{B_0} &= x|_{B_0} = y|_{B_0} = 0. \end{aligned}$$

It is clear that $x, y, z \in S_\varphi$ and $z = \frac{1}{2}(x + y)$. Denote

$$b = \sup\{t \geq 0: \varphi(t) \leq 1/\mu(B_1)\}$$

and pick $0 < \varepsilon < \gamma \cdot b^{-1}$ and $u \in B_\varphi$ with $\|u - x\|_\varphi < \varepsilon$. Then

$$\int_{B_1} \varphi((u - x)/\varepsilon) d\mu < 1$$

and if

$$C = \left\{ t \in B_1: \varphi\left(\frac{u(t) - x(t)}{\varepsilon}\right) \leq \frac{1}{\mu(B_1)} \right\} = \{t \in B_1: |u(t) - (a_0 + \gamma)| \leq \varepsilon b\}$$

then we have $\mu(C) > 0$. Therefore, as $\varepsilon b < \gamma$, we get $u(t) > a_0$, $t \in C$, and $\int_{B_1} \varphi(u) d\mu > 0$. Hence condition (i) of (*) holds. To establish (ii) of (*), assume (A2) and take $\delta > 0$ arbitrary. Let $p = \min\{m \in \mathbb{N}: F(m) \geq 1\}$ and $\lambda \in (0, 1]$ such that

$$\sum_{n=2}^{q-1} \int_{B_n} \varphi(g) d\mu + \sum_{n=q}^p \int_{B_n} \varphi([1 + \lambda c_p(|g|)]g) d\mu + \sum_{n>p} \int_{B_n} \varphi(g) d\mu = 1.$$

Define $w \in L^\varphi(\mu)$ as

$$\begin{aligned} w|_{B_1} &= a_0; & w|_{B_n} &= g, & 2 \leq n < q \text{ and } n > p; \\ w|_{B_n} &= [1 + \lambda c_p(|g|)]g, & q \leq n \leq p; & & w|_{B_0} &= 0 \end{aligned}$$

Then $\|w\|_\varphi = I_\varphi(w) = 1$ and, since $\delta^{-1} \cdot \lambda c_p(|g(t)|) < 1$, $t \in B_n$, $q \leq n \leq p$, we have

$$I_\varphi(\delta^{-1}(w - z)) = \sum_{n=q}^p \int_{B_n} \varphi(g \delta^{-1} \lambda \cdot c_p(|g|)) d\mu \leq \sum_{n=q}^p \int_{B_n} \varphi(g) d\mu \leq I_\varphi(g) < 1,$$

i.e., $\|w - z\|_\varphi \leq \delta$. Hence condition (ii) of (*) holds and $L^\varphi(\mu)$ satisfies (*).

(B) Assume the hypothesis of Lemma 2 and prove that (A1) and (A2) are fulfilled. First, note that $a_0 < \tau$. Indeed, if $a_0 = \tau$, then (2) works, $|f(t)| \leq a_0$ a.e., and as $\int_\Omega \varphi(\lambda f) d\mu = +\infty$, $\forall \lambda > 1$, necessarily

$$\mu(\{t \in \Omega: |f(t)| \geq a_0 - 1/n\}) > 0, \quad n \geq 1,$$

a contradiction since f is nonterminal. Define $L_0 = \{t \in \Omega: |f(t)| \leq a_0\}$ and for $n \geq 1$

(α) If (1) holds, $U_n = \{t \in \Omega: \tau_n \leq |f(t)| < \tau_{n+1}\}$ with $\tau_n > 0$, $\varphi(\tau_n) = n$, $n \geq 1$, and $L_n = \{t \in \Omega: (1/(n+1)) \leq \varphi(f(t)) < 1/n\}$.

(β) If (2) holds, $U_n = \emptyset$ and $L_n = \{t \in \Omega: (\tau - \beta - a_0)/(n+1) < |f(t)| - a_0 \leq (\tau - \beta - a_0)/n\}$ with $0 < \beta < \tau$ and $|f(t)| \leq \tau - \beta$ a.e.

Then $0 \leq \mu(U_n) < \infty$, $0 \leq \mu(L_n) < \infty$, $n \geq 1$, $0 \leq \mu(L_0) \leq \infty$, and at least one of the following facts is verified:

$$(a) \sum_{n \geq 1} \int_{U_n} \varphi(\lambda f) d\mu = \infty, \quad \forall \lambda > 1; \quad (b) \sum_{n \geq 0} \int_{L_n} \varphi(\lambda f) d\mu = \infty, \quad \forall \lambda > 1.$$

(I) Assume that (a) occurs. Then $\lim_{t \rightarrow \tau^-} \varphi(t) = +\infty$. Suppose that $\mu(U_1) > 0$ (if $\mu(U_1) = 0$, take the first U_m with $\mu(U_m) > 0$). Choose $\gamma > 0$ such that $\int_{U_1} \varphi(a_0 + \gamma) d\mu + \sum_{n \geq 2} \int_{U_n} \varphi(f) d\mu \leq 1$ and define $B_n = U_n$, $n \geq 1$, and

$g = f$. Then (A1) is satisfied. To establish (A2), fix $\delta > 0$ and consider the following two cases:

Case 1. $\tau < \infty$. Pick $q \in \mathbb{N}$ with $q \geq 2$ and $\tau \leq (1 + \delta)\tau_q$. For $m \in \mathbb{N}$ put $c_m(0) = 0$ and, if $t > 0$, let $c_m(t)$ be such that $(1/m)t + (1 - 1/m)\tau = (1 + c_m(t))t$, i.e., $c_m(t) = (\tau/t - 1)(1 - 1/m)$. If $t \in [\tau_q, \tau)$ then $c_m(t) = (\tau/t - 1)(1 - 1/m) < \delta$, i.e., $c_m(|f(t)|) \cdot \delta^{-1} < 1$, $t \in U_i = B_i$, $i \geq q$. Also $F(m) < \infty$, for every $m \in \mathbb{N}$, but $f(m)_\uparrow + \infty$, whereas $m \rightarrow +\infty$. Hence (A2) holds.

Case 2. $\tau = \infty$. Take $q = 2$, $0 < h^{-1} < \delta$ and define $c_m(t) = h^{-1}$, $t \geq 0$, $m \in \mathbb{N}$. Then $\delta^{-1} \cdot c_m(|f(t)|) < 1$, $t \in U_i = B_i$, $i \geq 2$, and

$$F(m) = \sum_{n=2}^m \int_{U_n} \varphi \left(\left(1 + \frac{1}{h} \right) f \right) d\mu + \sum_{n>m} \int_{U_n} \varphi(f) d\mu, \quad m \geq 2,$$

satisfies $F(m) < \infty$, $m \geq 1$, but $F(m)_\uparrow + \infty$ whereas $m \rightarrow \infty$. Hence (A2) is fulfilled.

(II) Assume that (b) is satisfied. Put $\sigma = \tau_1$ if (1) holds, and put $\sigma = \tau - \beta$, if (2) holds. Then $0 < \sigma < \tau$ and $|f(t)| \leq \sigma$, $\forall t \in \bigcup_{n \geq 0} L_n$. As (Ω, Σ, μ) is σ -finite, we have a pairwise disjoint partition $\bigcup_{n \geq 1} B_n = \bigcup_{n \geq 0} L_n$, with $0 \leq \mu(B_n) < +\infty$ and $\mu(B_1) > 0$. (Note that $\mu(\bigcup_{n \geq 0} L_n) > 0$. In fact $\mu(\bigcup_{n \geq 0} L_n) = +\infty$.) Now $\sum_{n \geq 2} \int_{B_n} \varphi(\lambda f) d\mu = \infty$, $\forall \lambda > 1$ (because $\sum_{n \geq 1} \int_{B_n} \varphi(\lambda f) d\mu = 0$, $\forall \lambda > 1$ and $\int_{B_1} \varphi(f \cdot \lambda_0) d\mu < \infty$ if $1 < \lambda_0 < \tau/\sigma$), and we can choose $0 < \gamma$ such that $\int_{B_1} \varphi(a_0 + \gamma) d\mu + \sum_{n \geq 2} \int_{B_n} \varphi(f) d\mu \leq 1$. Putting $g = f$, (A1) is satisfied. For condition (A2), given $\delta > 0$, take $q = 2$ and $0 < h^{-1} < \delta$ such that $(1 + h^{-1})\sigma < \tau$ and define $c_m(t) = h^{-1}$, $t \geq 0$, $m \in \mathbb{N}$. $\square$

Theorem 3. Let (Ω, Σ, μ) be a positive measure space and φ an Orlicz function such that $\lim_{x \rightarrow \tau^-} \varphi(x) = +\infty$. Then the following are equivalent (tfae):

- (1) B_φ is stable;
- (2) $I_\varphi(f) = 1$ for each $f \in S_\varphi$.

Proof. $1 \Rightarrow 2$. If there exists $f \in S_\varphi$ with $I_\varphi(f) < 1$, by Lemma 2, $L^\varphi(\mu)$ satisfies (*) and, by Lemma 1, B_φ is not stable.

$2 \Rightarrow 1$. Apply the proof of [2, Proposition 3]. $\square$

Theorem 4. Let φ be a finite Orlicz function and (Ω, Σ, μ) a positive measure space. Then tfae:

- (1) B_φ is stable;
- (2) $I_\varphi(f) = 1$ for every $f \in S_\varphi$;
- (3) $E^\varphi(\mu) = L^\varphi(\mu)$.

Proof. $1 \Leftrightarrow 2$. It is proved in Theorem 3.

$2 \Rightarrow 3$. If $E^\varphi(\mu) \neq L^\varphi(\mu)$, there exists $f \in S_\varphi$ such that $I_\varphi(f) < 1$. (See, e.g., the proof of $1 \Rightarrow 2$ of [2, Proposition 9].)

$3 \Rightarrow 2$. This is an easy consequence of the dominated convergence theorem. $\square$

Theorem 5. Let $\varphi(\tau) < \infty$ and (Ω, Σ, μ) a positive measure space. Then tfae:

- (1) B_φ is stable.

- (2) One of the following conditions holds: (i) $\mu(\Omega) < +\infty$; (ii) $\mu(\Omega) = +\infty$, $a_0 = \tau$, and $L^\varphi(\mu) \cong L^\infty(\mu)$ (isometry); (iii) $\mu(\Omega) = +\infty$, $0 = a_0 < \tau$, and $I_\varphi(f) = 1$ for every nonterminal $f \in S_\varphi$.
- (3) $I_\varphi(f) = 1$ for every nonterminal $f \in S_\varphi$.

Proof. $1 \Rightarrow 2$. Suppose that $\mu(\Omega) = +\infty$. If $a_0 = \tau$, then $L^\varphi(\mu)$ is trivially isometric to $L^\infty(\mu)$. Let $0 \leq a_0 < \tau$. By Lemma 2 and Lemma 1, every nonterminal $f \in S_\varphi$ verifies $I_\varphi(f) = 1$. If $0 < a_0$ then the function $f(t) = a_0$, $\forall t \in \Omega$, should be a nonterminal $f \in S_\varphi$ with $I_\varphi(f) = 0$. Hence $a_0 = 0$.

$2 \Rightarrow 3$. Trivially (iii) implies (3) and, if (ii) holds, every $f \in S_\varphi$ is terminal. Assume, finally, that $\mu(\Omega) < \infty$ and let $f \in S_\varphi$ be nonterminal. Then $|f(t)| \leq \tau - \beta$ for some $0 < \beta < \tau$ and, if $1 < \lambda_0 < \tau/(\tau - \beta)$, we have

$$1 \leq I_\varphi(\lambda_0 \cdot f) = \int_\Omega \varphi(\lambda_0 \cdot f) d\mu \leq \int_\Omega \varphi(\tau) d\mu = \varphi(\tau) \cdot \mu(\Omega) < +\infty.$$

Therefore, by the dominated convergence theorem, we have $I_\varphi(f) = 1$.

$3 \Rightarrow 1$. Let $x, y, z \in S_\varphi$ be such that $z = \frac{1}{2}(x + y)$. We prove that, given V^x, V^y neighborhoods of x and y in B_φ , $\frac{1}{2}(V^x + V^y)$ is a neighborhood of z in B_φ . Without loss of generality, assume that $z(t) \geq 0$, $t \in \Omega$.

If $I_\varphi(z) = 1$, apply the proof of Proposition 3 of [2].

Let $I_\varphi(z) < 1$. Then z is terminal. If $a_0 = \tau$ then $L^\varphi(\mu) \cong L^\infty(\mu)$ (isometry). Thus B_φ is stable because $L^\infty(\mu)$ has the 3.2.I.P. [1, p. 195]. Suppose that $a_0 < \tau$. Fix $\varepsilon > 0$ and take $\frac{1}{2} < \lambda < 1$ such that $\|x - \bar{x}\|_\varphi \leq \frac{1}{2}\varepsilon$, $\|y - \bar{y}\|_\varphi \leq \frac{1}{2}\varepsilon$, where $\bar{x} = \lambda x + (1 - \lambda)y$ and $\bar{y} = (1 - \lambda)x + \lambda y$. Clearly $z = \frac{1}{2}(\bar{x} + \bar{y})$ and $I_\varphi(\bar{x}) < 1$, $I_\varphi(\bar{y}) < 1$, because for each $\rho \in (0, 1)$, if $f_\rho = \rho x + (1 - \rho)y$, we have $I_\varphi(f_\rho) < 1$. Indeed, for each $\rho \in (0, 1)$, there exists $\nu \in (0, 1]$ such that $f_\rho = \nu z + (1 - \nu)x$ or $f_\rho = \nu z + (1 - \nu)y$. If, for instance, $f_\rho = \nu z + (1 - \nu)x$, by convexity of I_φ we have

$$I_\varphi(f_\rho) = I_\varphi(\nu z + (1 - \nu)x) \leq \nu I_\varphi(z) + (1 - \nu)I_\varphi(x) < \nu + (1 - \nu) = 1.$$

Let $0 < \alpha < \min\{1 - I_\varphi(\bar{x}), 1 - I_\varphi(\bar{y})\}$. Choosing suitable representative functions, assume that $0 \leq z(t) \leq \tau$, $\bar{x}(t), \bar{y}(t), x(t), y(t) \in [-\tau, \tau]$, $z(t) = \frac{1}{2}(\bar{x}(t) + \bar{y}(t))$, $\bar{x}(t) = \lambda x(t) + (1 - \lambda)y(t)$, and $\bar{y}(t) = (1 - \lambda)x(t) + \lambda y(t)$, $\forall t \in \Omega$. For $0 \leq s \leq \tau$, define

$$A_s = \{t \in \Omega: |\bar{x}(t)| \leq \tau - s \leq |\bar{y}(t)|\}, \quad B_s = \Omega \setminus A_s, \quad C = \{t \in \Omega: z(t) = \tau\}.$$

By the above choice of the representative functions, it is clear that $C \subseteq \{t \in \Omega: \bar{x}(t) = \tau\} \cap \{t \in \Omega: \bar{y}(t) = \tau\}$. Further, we have that $\bar{x}(t) = \tau$ iff $x(t) = \tau = y(t)$ iff $\bar{y}(t) = \tau$. Thus $C = \{t \in \Omega: \bar{x}(t) = \tau\} = \{t \in \Omega: \bar{y}(t) = \tau\}$. By a similar argument, we get that $\{t \in \Omega: \bar{x}(t) = -\tau\} = \emptyset = \{t \in \Omega: \bar{y}(t) = -\tau\}$. Hence $\lim_{s \downarrow 0} \mu(B_s \setminus C) = 0$ and

$$\begin{aligned} \lim_{s \downarrow 0} \int_{B_s} [\varphi(\tau) - \varphi(\bar{x}(t))] d\mu &= 0 = \lim_{s \downarrow 0} \int_{B_s} [\varphi(\tau) - \varphi(\bar{y}(t))] d\mu, \\ \int_{B_s} \varphi(\bar{x}(t)) d\mu &< 1 - \alpha - \frac{\nu}{2} - \int_{A_s} \varphi(\bar{x}(t)) d\mu, \\ \int_{B_s} \varphi(\bar{y}(t)) d\mu &< 1 - \alpha - \frac{\nu}{2} - \int_{A_s} \varphi(\bar{y}(t)) d\mu \end{aligned}$$

for some $\nu > 0$ and every $0 \leq s \leq \tau$. Choose $0 < r < \tau$ such that

$$\int_{B_r} [\varphi(\tau) - \varphi(\bar{x}(t))] d\mu < \frac{\nu}{2} > \int_{B_r} [\varphi(\tau) - \varphi(\bar{y}(t))] d\mu.$$

Then

$$\mu(B_r) \cdot \varphi(\tau) \leq 1 - \alpha - \int_{A_r} \varphi(\bar{x}(t)) d\mu, \quad \mu(B_r) \cdot \varphi(\tau) \leq 1 - \alpha - \int_{A_r} \varphi(\bar{y}(t)) d\mu.$$

Since $|\bar{x}(t)| \leq \tau - r \geq |\bar{y}(t)|$ for each $t \in A_r$, we can find $\gamma > 1$ such that

$$\int_{A_r} \varphi(\gamma \bar{x}) d\mu \leq \int_{A_r} \varphi(\bar{x}) d\mu + \alpha = \alpha_1, \quad \int_{A_r} \varphi(\gamma \bar{y}) d\mu \leq \int_{A_r} \varphi(\bar{y}) d\mu + \alpha = \alpha_2.$$

Indeed, necessarily $\|\bar{x} \cdot \chi_{A_r}\|_\varphi < 1$ because if $\|\bar{x} \cdot \chi_{A_r}\|_\varphi = 1$ then, as $\bar{x} \cdot \chi_{A_r}$ is nonterminal, we would have by hypothesis that $\int_{A_r} \varphi(\bar{x}) d\mu = 1$. Therefore, there exists $\lambda > 1$ such that $\|\lambda \bar{x} \cdot \chi_{A_r}\|_\varphi \leq 1$ and $H(\lambda) = \int_{A_r} \varphi(\lambda \bar{x}) d\mu < +\infty$. As $\lim_{\lambda \downarrow 1} H(\lambda) = H(1)$, we can choose $\gamma > 1$ such that $H(\gamma) \leq H(1) + \alpha$. Analogously for $\bar{y}$.

Since, with the hypothesis $\tau < \infty$, the canonical immersion $L^\varphi(\mu) \rightarrow L^\infty(\mu)$ is continuous, there exists $\rho > 0$ such that $\|f\|_\infty < r$ if $\|f\|_\varphi < \rho$. Take $\delta = \min\{(1-1/\gamma)\alpha_1, (1-1/\gamma)\alpha_2, \rho, \varepsilon/2\}$ and $w \in B_\varphi$ such that $\|w - z\|_\varphi < \delta/2$. Choosing a suitable representative function, assume that $w(t) \in [-\tau, \tau]$ and $|w(t) - z(t)| < r/2$, $\forall t \in \Omega$. We want to find $u, v \in B_\varphi$ such that $w = \frac{1}{2}(u+v)$ and $\|x - u\|_\varphi < \varepsilon$, $\|y - v\|_\varphi < \varepsilon$. Define, $u, v \in L^\varphi(\mu)$ as follows:

(I) If $t \in A_r$, $u(t) = \bar{x}(t) + w(t) - z(t)$ and $v(t) = \bar{y}(t) + w(t) - z(t)$;

(II) If $t \in B_r$, we have four cases:

- (1) $w(t) \geq z(t)$ and $\bar{x}(t) \geq z(t)$: $u(t) = \inf[\bar{x}(t) + w(t) - z(t), \tau]$, $v(t) = 2w(t) - u(t)$.
- (2) $w(t) \geq z(t)$ and $\bar{y}(t) > z(t)$: $v(t) = \inf[\bar{y}(t) + w(t) - z(t), \tau]$, $u(t) = 2w(t) - v(t)$.
- (3) $w(t) < z(t)$ and $\bar{x}(t) \leq z(t)$: $u(t) = \sup[\bar{x}(t) + w(t) - z(t), -\tau]$, $v(t) = 2w(t) - u(t)$.
- (4) $w(t) < z(t)$ and $\bar{y}(t) < z(t)$: $v(t) = \sup[\bar{y}(t) + w(t) - z(t), -\tau]$, $u(t) = 2w(t) - v(t)$.

Then $w = \frac{1}{2}(u+v)$, $|u(t)| \leq \tau \geq |v(t)|$, and $|u(t) - \bar{x}(t)| \leq 2|w(t) - z(t)| \geq |v(t) - \bar{y}(t)|$ for each $t \in \Omega$. Therefore $\|x - u\|_\varphi \langle \varepsilon \rangle \|y - v\|_\varphi$. We claim that $u, v \in B_\varphi$. Since $I_\varphi(s) \leq \|s\|_\varphi$, provided $\|s\|_\varphi \leq 1$, we have

$$I_\varphi\left(\frac{\gamma}{\gamma-1} \cdot 2(w-z)\right) \leq \left\|\frac{\gamma}{\gamma-1} \cdot 2(w-z)\right\|_\varphi \leq \alpha_i, \quad i = 1, 2.$$

Hence

$$\begin{aligned} I_\varphi(u) &= \int_{B_r} \varphi(u) d\mu + \int_{A_r} \varphi\left[\frac{1}{\gamma} \gamma \bar{x} + \left(1 - \frac{1}{\gamma}\right) \frac{\gamma}{\gamma-1} (u - \bar{x})\right] d\mu \\ &\leq \varphi(\tau) \cdot \mu(B_r) + \frac{1}{\gamma} \int_{A_r} \varphi(\gamma \bar{x}) d\mu + \left(1 - \frac{1}{\gamma}\right) \int_{A_r} \varphi\left(\frac{\gamma}{\gamma-1} \cdot 2(w-z)\right) d\mu \\ &\leq \varphi(\tau) \cdot \mu(B_r) + \frac{1}{\gamma} \left[\int_{A_r} \varphi(\bar{x}) d\mu + \alpha\right] + \left(1 - \frac{1}{\gamma}\right) \left[\int_{A_r} \varphi(\bar{x}) d\mu + \alpha\right] \\ &= \varphi(\tau) \cdot \mu(B_r) + \int_{A_r} \varphi(\bar{x}) d\mu + \alpha \leq 1. \end{aligned}$$

Analogously $I_\varphi(v) \leq 1$ and the proof is complete. $\square$

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