A CHARACTERIZATION OF FINITELY JUNCTIONED CONTINUA

EDWIN DUDA AND PAWEŁ KRUPSKI

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ABSTRACT. Metric continua which are k -junctioned are defined and characterized. Furthermore such continua are shown to be invariant under open or monotone mappings.

A metric continuum is said to be k -junctioned if it is an inverse limit of graphs each of which has at most k branch points, with surjective bonding maps. A continuum is called *finitely junctioned* if it is k -junctioned for some integer $k \geq 0$.

Finitely junctioned continua were considered in relation to homogeneity in [2, 4, 7].

It follows from the definition that a continuum is 0-junctioned if and only if it is either chainable or circularly chainable. In this note we obtain an analogous characterization for all finitely junctioned continua (Theorem 1). It follows, for example, that if each proper subcontinuum of a nondegenerate metric continuum is chainable, then the continuum is 1-junctioned. The result is the best possible (see [5]).

Another interesting and easy consequence of Theorem 1 is the invariance of k -junctionedness under monotone and open mappings. It is known that a monotone image of a chainable continuum is chainable [1] and a monotone image of a circularly chainable continuum is circularly chainable [3]. We extend these results to k -junctioned continua (Theorem 2). An open image of a chainable continuum is chainable, by [8], but the old problem of whether an open image of a circularly chainable continuum has to be 0-junctioned remains, in general, unsolved (positive answers are known for solenoids [6] or under the additional assumption that the range space is decomposable [3]). It follows from our Theorem 3 that an open image of a circularly chainable continuum is 1-junctioned.

An element of a cover of a space is called a *junction element* if it meets at least three other elements of the cover. A collection of open subsets $U_1, U_2, \dots, U_n$ of a space X is a *chain* (*circular chain*) in X if $U_i \cap U_j \neq \emptyset \Leftrightarrow |i - j| \leq 1$ ($|i - j| \leq 1$ or $\{i, j\} = \{1, n\}$). We will use an equivalent definition of k -junctioned continua in terms of covers. A metric nondegenerate continuum X

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is k -junctioned if and only if for every $\varepsilon > 0$ there exists a finite open ε -cover of X of order one with at most k -junction elements. A metric continuum X is *chainable* (*circularly chainable*) if for every $\varepsilon > 0$ there exists an ε -chain (-circular chain) covering X . All mappings are assumed to be continuous.

Theorem 1. *A metric continuum X is k -junctioned, where $k > 0$, if and only if X contains a k -element subset P such that every subcontinuum of $X - P$ is chainable.*

Proof. The “only if” part has been proved in [4, Theorem 7]. For the sake of completeness we repeat here an idea of its proof. We may assume that X is not m -junctioned for any $m < k$. Therefore, there exist finite open covers $\mathcal{E}_n$ of X such that $\text{mesh } \mathcal{E}_n \rightarrow 0$ and $\mathcal{E}_n$ contains exactly k junction elements $C_n^1, \dots, C_n^k$ for every $n = 1, 2, \dots$. Define subsequences $\alpha_1, \dots, \alpha_k$ of natural numbers and points $p_1, \dots, p_k$ of X by induction: α_1 is a sequence $(n_1, n_2, \dots)$ such that the sets $C_{n_1}^1, C_{n_2}^1, \dots$ converge to a point p_1 ; for $1 \leq i \leq k-1$, α_{i+1} is a subsequence $(a_1, a_2, \dots)$ of α_i such that the sets $C_{a_1}^{i+1}, C_{a_2}^{i+1}, \dots$ converge to a point p_{i+1} . Put $P = \{p_1, \dots, p_k\}$ and $\alpha_k = (a_1, a_2, \dots)$. If K is a subcontinuum of $X - P$, then for sufficiently large m all the junction elements $C_{a_m}^1, \dots, C_{a_m}^k$ of the cover $\mathcal{E}_{a_m}$ are disjoint with K . Therefore the family $\{C \in \mathcal{E}_{a_m} : C \cap K \neq \emptyset\}$ is a chain in X of arbitrarily small mesh covering K (it cannot be a circular chain, for $k > 1$). It means that K is chainable.

Suppose now a metric continuum X contains a subset $P = \{p_1, \dots, p_k\}$ such that every subcontinuum of $X - P$ is chainable. Let $\varepsilon > 0$ be a positive number such that if $k > 1$ then $\varepsilon < \min\{\text{dist}(p_i, p_j) : i, j = 1, \dots, k\}$. Consider open $\varepsilon/2$ -neighborhoods $B_1, \dots, B_k$ of points $p_1, \dots, p_k$, respectively. For every component C of the compact set $Z = X - \bigcup_{i=1}^k B_i$ denote by $\mathcal{U}_C$ an $\varepsilon/2$ -chain in X covering C . There is an open set $V_C \subset X$ such that $C \subset V_C \subset \overline{V_C} \subset \bigcup \mathcal{U}_C$ and $\text{bd } V_C \cap Z = \emptyset$. Choose a finite number of sets $V_{C_1}, \dots, V_{C_n}$ covering Z . The sets $W_i = V_{C_i} \cap Z$ are closed and open in Z for $i = 1, \dots, n$. Therefore the sets $T_1 = W_1$ and $T_i = W_i - (W_1 \cup \dots \cup W_{i-1})$, for $i = 2, \dots, n$, are closed and open in Z , disjoint and cover Z . Now, the sets $U \cap T_i$, where $U \in \mathcal{U}_C$, are open in X unless

$$(*) \quad U \cap T_i \cap \text{bd } B_j \neq \emptyset \quad \text{for some } j.$$

Put, for every $j = 1, \dots, k$,

$$G_j = B_j \cup \bigcup \{U : U \in \mathcal{U}_{C_i}, U \text{ satisfies } (*) \text{ and } i = 1, \dots, n\}$$

and define

$$\mathcal{G} = \{G_1, \dots, G_k\} \cup \{U : U \in \mathcal{U}_{C_i}, U \text{ does not satisfy } (*) \text{ and } i = 1, \dots, n\}.$$

It follows from the construction that $\mathcal{G}$ is an open ε -cover of X of order one and the only junction elements of $\mathcal{G}$ can be $G_1, \dots, G_k$. The proof of Theorem 1 is complete.

Theorem 2. *If f is a monotone mapping of a k -junctioned continuum X onto a nondegenerate Hausdorff space Y , then the Y is a k -junctioned continuum.*

Proof. The result is known for $k = 0$ [1, 3]. Suppose $k > 0$ and let P be a k -element subset of X satisfying Theorem 1. In view of Theorem 1, it is enough

to observe that every subcontinuum K of $Y - f(P)$ is chainable. Indeed, $f^{-1}(K)$ is a chainable subcontinuum of $X - P$ by Theorem 1 and the mapping f restricted to $f^{-1}(K)$ is monotone; so K is chainable by Bing's result [1].

It is proved in [3, Theorem 3.2] that if every proper subcontinuum of a metric continuum X is chainable, then so is every proper subcontinuum of an open image of X .

Actually, the proof gives the following statement.

Lemma [3]. *If f is an open mapping of a metric continuum X onto a Hausdorff space and K is a proper subcontinuum of Y such that a component of $f^{-1}(K)$ is a chainable continuum, then the continuum K is chainable.*

Theorem 3. *Let f be an open mapping of a k -junctioned continuum X onto a nondegenerate Hausdorff space Y . If $k = 0$ then Y is a 1-junctioned continuum. If $k > 0$ then Y is a k -junctioned continuum.*

Proof. In order to prove the case $k = 0$ pick a point $p \in Y$, a subcontinuum $K \subset Y - \{p\}$, and a component C of $f^{-1}(K)$. The component C , being a proper subcontinuum of X , is chainable and lemma implies K is chainable. By Theorem 1 the continuum Y is 1-junctioned. The proof of the case $k > 0$ is similar.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF MIAMI, CORAL GABLES, FLORIDA 33124

MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCLAW, PL GRUNWALDSKI 2/4, 50-384 WROCLAW, POLAND