$\mathbf{R}^{1|1}$ -SUPERGROUP ACTIONS AND SUPERDIFFERENTIAL EQUATIONS

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ABSTRACT. The problem of posing and solving ordinary differential equations on supermanifolds is addressed from the point of view of Lie's theory. It is shown that no nonsingular, nondegenerate, odd supervector field can have a Lie supergroup action of $\mathbf{R}^{1|1}$ as its flow. It is also shown that the class of integrable supervector fields goes far beyond homogeneity and rectifiability. The obstructions for the integral flow to be an $\mathbf{R}^{1|1}$ -action are given by the Lie superbracket of the field with itself and the Lie bracket of its homogeneous components.

INTRODUCTION

The C^∞ -solution of the C^∞ ODE defined by a given vector field X on a smooth manifold M yields a Lie group action of $\mathbf{R}$ on M . The point of this paper is to determine whether or not such a rich structure is still present when the ODE problem is formulated in the category of supermanifolds. The main result is that not every supervector field can have an $\mathbf{R}^{1|1}$ -action as its integral flow; the obstruction resembles Frobenius theorem: for nonsingular supervector fields having a normal form defined on $\mathbf{R}^{1|1}$ (a class that includes the generic, homogeneous, and rectifiable cases studied in [8, 5]), it is measured by the Lie superbracket of the field with itself and the Lie superbracket of its homogeneous components. In particular, there are lots of nonhomogeneous superfields whose flow define an $\mathbf{R}^{1|1}$ -action when integrated. Thus, the class of "integrable" supervector fields goes far beyond the rectifiable (always homogeneous) cases.

1. GENERAL BACKGROUND

The problem of posing and solving ordinary differential equations on supermanifolds was first treated in [8]. It is shown there that under a generic hypothesis (called *weak nondegeneracy*; cf. §3), a nonsingular homogeneous supervector field may be brought to a normal form: ∂_{x_1} if the field is even,

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or $\partial_{\xi^1} + \xi^1 Y$ if the field is odd, Y being an even field. When Y itself satisfies the same generic hypothesis, the odd normal form reduces even further to $\partial_{\xi^1} + \xi^1 \partial_{x^1}$. The case when $Y = 0$ is also special; it corresponds to weakly nondegenerate odd fields with vanishing Lie-self-superbracket. The three simplest normal forms single out the class of *rectifiable* supervector fields (cf. [5]). In particular, the integration problem for the ODE's defined by such fields is reduced to a problem in $\mathbf{R}^{1|1}$.

Now, every supervector field X on a superdomain $\mathbf{M}$ gives rise to an ODE with prescribed *initial data*. In analogy to the C^∞ -theory, a solution is given in terms of *supertime parameters*, i.e., a set of local coordinates on some sub-supermanifold $\mathbf{T}$ of $\mathbf{R}^{1|1}$ containing $0 \in \mathbf{R}$. To solve the ODE determined by X means to find a supermanifold morphism

$$(1.1) \quad \Gamma: \mathbf{T} \times \mathbf{M} \rightarrow \mathbf{M}$$

such that

$$(1.2) \quad ev|_{t=t_0} \circ \tilde{D} \circ \Gamma^\sharp = ev|_{t=t_0} \circ \Gamma^\sharp \circ X,$$

subject to the initial condition

$$(1.3) \quad \Gamma \circ (C_0 \times \text{id}) = \text{id}.$$

The equality (1.2) is understood as superderivations of the sheaf of superfunctions on $\mathbf{M}$; $\Gamma^\sharp$ pulls back superfunctions on $\mathbf{M}$ to $\mathbf{T} \times \mathbf{M}$ via Γ ; $\tilde{D}$ is the lift to $\mathbf{T} \times \mathbf{M}$ of a preferred superfield D on $\mathbf{T} \subset \mathbf{R}^{1|1}$ (cf. §4). It is defined by the conditions, $\tilde{D} \circ \pi_1^\sharp = \pi_1^\sharp \circ D$ and $\tilde{D} \circ \pi_2^\sharp = 0$; π_1 and π_2 being the projections of $\mathbf{T} \times \mathbf{M}$ into the corresponding factors. Finally, we have used the suggestive symbol ' $ev|_{t=t_0}$ ' to denote the pull-back algebra map resulting from the inclusion $C_{t_0} \times \text{id}: \mathbf{M} \rightarrow \mathbf{T} \times \mathbf{M}$. Here, C_{t_0} is the constant map defined by the algebra morphism $f \mapsto \tilde{f}(t_0)1_{\mathbf{M}}$, from superfunctions, f on $\mathbf{T}$, to superfunctions on $\mathbf{M}$ (cf. [3, 4, 1]).

On the other hand, Lie supergroup actions have been studied in [1]. The supermanifold $\mathbf{R}^{1|1}$ has a naturally defined sumlike morphism,

$$(1.4) \quad \sigma: \mathbf{R}^{1|1} \times \mathbf{R}^{1|1} \rightarrow \mathbf{R}^{1|1},$$

which gives it the structure of an additive supergroup. More generally, it was pointed out in [6] that there is also a natural productlike morphism $\mu: \mathbf{R}^{1|1} \times \mathbf{R}^{1|1} \rightarrow \mathbf{R}^{1|1}$ such that, when combined with σ , make $\mathbf{R}^{1|1}$ to play the role of $\mathbf{R}$ in the "super" category, as there is a local identification between the structural sheaf of every supermanifold and the maps from the supermanifold into $\mathbf{R}^{1|1}$ (cf. [6]). In all, this gives good reasons for choosing $\mathbf{R}^{1|1}$ as home for the integration parameters.

2. ON THE SUPERGROUP ACTION OF $\mathbf{R}^{1|1}$

From now on, we shall assume that $\mathbf{T}$ is actually $\mathbf{R}^{1|1}$. Following [1], (1.1) is a Lie supergroup action if, in addition to (1.3), the following equation is satisfied:

$$(2.1) \quad \Gamma \circ (\sigma \circ (\pi_1 \times \pi_2) \times \pi_3) = \Gamma \circ (\pi_1 \times \Gamma \circ (\pi_2 \times \pi_3)).$$

Here, π_i denotes the projection morphism onto the i th factor of the product $\mathbf{R}^{1|1} \times \mathbf{R}^{1|1} \times \mathbf{M}$. Now, in combining the morphism Γ with the local problem

of actually integrating equation (1.2), a coordinate expression for Γ is needed. Let $\mathbf{M}$ be an (m, n) -dimensional superdomain with coordinates $\{y^i; \theta^\mu\}$, and let $\{t; \tau\}$ be a global set of coordinates on $\mathbf{R}^{1|1}$. We shall write

$$(2.2a) \quad \Gamma^\# y^i = \gamma_0^i + \sum_\nu \gamma_\nu^i p_1^\# \tau p_2^\# \theta^\nu + \sum_{\mu < \nu} \gamma_{\mu\nu}^i p_2^\# \theta^\mu p_2^\# \theta^\nu + \dots,$$

$$(2.2b) \quad \begin{aligned} \Gamma^\# \theta^\rho &= g_0^\rho p_1^\# \tau + \sum_\nu g_\nu^\rho p_2^\# \theta^\nu + \sum_{\mu < \nu} g_{\mu\nu}^\rho p_1^\# \tau p_2^\# \theta^\mu p_2^\# \theta^\nu \\ &+ \sum_{\lambda < \mu < \nu} g_{\lambda\mu\nu}^\rho p_2^\# \theta^\lambda p_2^\# \theta^\mu p_2^\# \theta^\nu + \dots \end{aligned}$$

where p_j is the projection onto the j th factor of the product $\mathbf{R}^{1|1} \times \mathbf{M}$. The conditions on Γ to define an action translate into conditions on the C^∞ coefficients appearing in this expansion. When the superdomain $\mathbf{M}$ has odd dimension bigger than two, the result is some complicated combination of functional and differential equations among the coefficients. Fortunately, for those superfields having a normal form defined on $\mathbf{R}^{1|1}$, the set of resulting relations are far more tractable.

2.1. Theorem. *Let $\mathbf{M}$ be a $(1, 1)$ -dimensional superdomain with coordinates $\{y; \theta\}$. Let $\{t; \tau\}$ be some global set of coordinates on $\mathbf{R}^{1|1}$ (e.g., those used in [1, Example 1.2]), and let*

$$(2.3) \quad \Gamma^\# y = \gamma_0 + \gamma_1 \tau \theta \quad \text{and} \quad \Gamma^\# \theta = g_0 \tau + g_1 \theta$$

be the local expression of Γ . Then Γ defines a (local) Lie supergroup action if and only if $\gamma_0(0, y) = y$, $g_1(0, y) = 1$, and for all t_1, t_2 , and y ,

$$(2.4) \quad \begin{aligned} \gamma_0(t_1, \gamma_0(t_2, y)) &= \gamma_0(t_1 + t_2, y), \\ \gamma_1(t_1, \gamma_0(t_2, y)) g_0(t_2, y) &= 0, \\ \gamma_1(t_1, \gamma_0(t_2, y)) g_1(t_2, y) &= \gamma_1(t_1 + t_2, y), \\ \partial_y \gamma_0(t_1, \gamma_0(t_2, y)) \gamma_1(t_2, y) &= \gamma_1(t_1 + t_2, y), \\ g_0(t_1, \gamma_0(t_2, y)) &= g_0(t_1 + t_2, y), \\ g_1(t_1, \gamma_0(t_2, y)) g_0(t_2, y) &= g_0(t_1 + t_2, y), \\ g_1(t_1, \gamma_0(t_2, y)) g_1(t_2, y) &= g_1(t_1 + t_2, y), \\ \partial_y g_0(t_1, \gamma_0(t_2, y)) \gamma_1(t_2, y) &= 0. \end{aligned}$$

Proof. The four equations in the left column are obtained from the condition $\{\text{LHS}(1.1)\}^\# y = \{\text{RHS}(1.1)\}^\# y$ and the four equations on the right follow from $\{\text{LHS}(1.1)\}^\# \theta = \{\text{RHS}(1.1)\}^\# \theta$. The computation is straightforward and only uses the fact that $\sigma^\# = \pi_1^\# + \pi_2^\#$ and if $\Phi = \pi_1 \times \Gamma \circ (\pi_2 \times \pi_3)$ then the pull-back of any smooth function $f \in C^\infty(\mathbf{R} \times M)$ is given by

$$\Phi^\# f = f \circ \tilde{\Phi} + (\gamma_1 \circ (\pi_2 \times \pi_3)) (\partial_y f \circ \tilde{\Phi}) \pi_2^\# \tau \pi_3^\# \theta,$$

where $\tilde{\Phi}$ (resp. $(\pi_2 \times \pi_3)$) denotes the underlying C^∞ map induced on points by Φ (resp. $\pi_2 \times \pi_3$) (cf. [3, 4, 1, 7]). On the other hand, if $\Psi = \sigma \circ (\pi_1 \times \pi_2) \times \pi_3$, its effect on a smooth function $f \in C^\infty(\mathbf{R} \times M)$ is simply

$\Psi^\sharp f = f \circ \tilde{\Psi}$. Finally, the conditions $\gamma_0(0, y) = y$ and $g_1(0, y) = 1$ follow from

$$\{\text{LHS}(1.3)\}^\sharp y = \{\text{RHS}(1.3)\}^\sharp y \quad \text{and} \quad \{\text{LHS}(1.3)\}^\sharp \theta = \{\text{RHS}(1.3)\}^\sharp \theta,$$

respectively. $\square$

The conditions on the coefficients may be further simplified. No attempt shall be made here to obtain the most economical set of equations, but note that γ_0 is an $\mathbf{R}$ -action, and once this action is known, the value of $g_0(t, y)$ is given by $g_0(0, \gamma_0(t, y))$. Thus, only $g_0(0, y')$ needs to be known. Similarly, $\gamma_1(t, y)$ may be determined from $\gamma_1(0, y)$, by means of either the third or the fourth equations in the left column in (2.4). Also, $g_0(t, y)$ may be determined from $g_1(t, y)$, or vice-versa, in view of the second equation on the right column: $g_1(t, y) g_0(0, y) = g_0(t, y)$. At any rate, no matter which functions we determine in terms of which others, the resulting set is consistent with the whole set of equations given by the theorem. The more restrictive equations, however, are those with vanishing RHS's; in particular, it follows from (2.4) that

$$(2.5) \quad \gamma_1(t, y) g_0(0, y) = 0 \quad \text{for all } t \text{ and } y.$$

When a nonsingular, supervector field is given and the corresponding differential equation is posed, the C^∞ functions appearing in (2.3) must be determined from the coefficients of the field. It turns out that equation (2.5) gives a useful criterion—to be read out from the field itself—for determining if the flow may or may not be a supergroup action of $\mathbf{R}^{1|1}$.

3. LEVEL OF DEGENERACY OF SUPERVECTOR FIELDS

The arguments in this section require some explicit reference to the structural sheaf of a given (m, n) -dimensional superdomain $\mathbf{M}$. We shall thus write $\mathbf{M} = (M, \mathcal{A}_M)$ and denote by $\mathcal{I}_M$ the sheaf of ideals generated by the odd subsheaf $(\mathcal{A}_M)_1 \subset \mathcal{A}_M$. The filtration, $\mathcal{A}_M = \mathcal{I}_M^0 \supset \mathcal{I}_M \supset \mathcal{I}_M^2 \supset \cdots \supset \mathcal{I}_M^n \supset \{0\}$, induces a filtration on any locally free sheaf $\mathcal{M}$ of $\mathcal{A}_M$ -modules; namely, $\mathcal{M} \supset \mathcal{M}^2 \supset \cdots \supset \mathcal{M}^n \supset \{0\}$. Locally, if $U \subset M$ is an open set on which $\mathcal{M}(U) \simeq \mathcal{A}_M(U)^r \oplus \mathcal{A}_M(U)^s$ — (r, s) being the rank of $\mathcal{M}$ —then $\mathcal{M}^k|_U = (\mathcal{I}_M^k|_U)^r \oplus (\mathcal{I}_M^k|_U)^s$. For the locally free sheaf of $\mathcal{A}_M$ -modules, $\mathcal{D}er \mathcal{A}_M$, the filtration may be actually seen in terms of a given set of coordinates $\{y^i, \theta^\mu\}$. In writing a supervector field in a form analogous to the morphism (2.2),

$$(3.1) \quad \begin{aligned} X = & \sum_i \left\{ A^i + \sum_\mu A_\nu^i \theta^\nu + \sum_{\mu < \nu} A_{\mu\nu}^i \theta^\mu \theta^\nu + \cdots \right\} \partial_{y^i} \\ & + \sum_\rho \left\{ B^\rho + \sum_\nu B_\nu^\rho \theta^\nu + \sum_{\mu < \nu} B_{\mu\nu}^\rho \theta^\mu \theta^\nu + \cdots \right\} \partial_{\theta^\rho}, \end{aligned}$$

the filtration degree corresponds to polynomial expressions of at least that degree in the odd variables as coefficients of the basis elements ∂_{y^i} and ∂_{θ^ρ} .

Definition. A nonzero germ $X_x \in \mathcal{D}er \mathcal{A}_M|_x$ has level of degeneracy k at the point x , if $X_x \equiv 0 \pmod{\mathcal{D}er \mathcal{A}_M^j|_x}$, for all $j < k$, and $X_x \not\equiv 0 \pmod{\mathcal{D}er \mathcal{A}_M^k|_x}$.

Thus, a supervector field X has level k at x if there is an open neighborhood U with local coordinates defined about x over which

$$(3.2a) \quad X = \sum A_{\mu_1 \dots \mu_k}^j \theta^{\mu_1} \dots \theta^{\mu_k} \partial_{y^j} + \sum B_{\mu_1 \dots \mu_k}^p \theta^{\mu_1} \dots \theta^{\mu_k} \partial_{\theta^p} \\ + \text{higher order terms,}$$

with

$$(3.2b) \quad \sum (A_{\mu_1 \dots \mu_k}^j)^2 + \sum (B_{\mu_1 \dots \mu_k}^p)^2 \neq 0.$$

A straightforward computation shows that the level is independent of the local coordinate expression. Level-zero fields are called *weakly nondegenerate* in [8]. The main result for such fields is

3.1. Theorem (Shander [8]). *Let X be a homogeneous supervector field of level zero at x . There exists a coordinate neighborhood U of x (with coordinates $\{x^i, \xi^\mu\}$), such that $X = \partial_{x^1}$ if X is even and $X = \partial_{\xi^1} + \xi^1 Y$ if X is odd, Y being an even supervector field of level ≥ 0 at x such that $[\partial_{\xi^1}, Y] = 0$.*

3.2. Remark. If $X = \partial_{\xi^1} + \xi^1 Y$, as in the statement, then $X \circ X = \frac{1}{2}[X, X] = \frac{1}{2}(1 + \xi^1 Y \xi^1)Y$. Hence, if $\xi^1 Y \xi^1 \neq 0$ then its filtration degree is at least two. In that case, $[X, X] = 0$ implies $Y = 0$. In fact, when $Y \xi^1 \neq 0$, the field X may not be further simplified so as to yield a normal form on $\mathbf{R}^{1|1}$. The simplest example is given by $X = (1 + \xi^1 \xi^2) \partial_{\xi^1}$.

3.3. Remark. When the even superfield Y in the second half of the statement has level zero itself, one may further conclude that $X = \partial_{\xi^1} + \xi^1 \partial_{x^1}$. In that case, X defines an onto endomorphism of $\mathcal{A}_M(U)$, i.e., X is nondegenerate as defined in [8]. Conversely, if $X: \mathcal{A}_M(U) \rightarrow \mathcal{A}_M(U)$ is odd and onto, a change of coordinates exists bringing X into the form $X = \partial_{\xi^1} + \xi^1 \partial_{x^1}$ (cf. [8]).

3.4. Remark. Theorem 3.1 gives the best possible results. For example, the field $X = \partial_{\theta^1} + \theta^1 \theta^2 \partial_{\theta^2}$ cannot be reduced to a field involving only one odd coordinate ξ^1 . One reason is that $Y = \theta^2 \partial_{\theta^2}$ is one of the irreducible normal forms for even superfields of level one. Also, nonhomogeneous fields cannot always be brought to a normal form involving only one even and one odd coordinates. An example is given by $X = \partial_{y^1} + \theta^2 \partial_{y^2}$. Again, the problem lies in the fact that $\theta^2 \partial_{y^2}$ is one of the irreducible normal forms for odd superfields of level one.

4. THE LOCAL ODE PROBLEM ON SUPERMANIFOLDS

We shall now consider those supervector fields of level zero that have a normal form on $\mathbf{R}^{1|1}$. In order to actually write the equations resulting from (1.2), one must first specify what the special section D of $\mathbf{R}^{1|1}$ must be. We claim, however, that in terms of the local coordinates $\{t, \tau\}$, there is no loss of generality in assuming that

$$(4.1) \quad D = \partial_t + \partial_\tau.$$

In fact, after writing the most general D in the form $D = (a + b\tau)\partial_t + (c + d\tau)\partial_\tau$, lifting it up to $\mathbf{R}^{1|1} \times \mathbf{M}$, and performing the evaluation at $t = t_0$, one finds that the coefficients b and d do not appear in the final expression. Therefore, the

information comes only from $a\partial_t + c\partial_\tau$, but then a simple change of coordinates puts D in the form (4.1).

In general, using the local expressions (2.2) and (3.1), together with D above, (1.2) translates into the following series of equations:

$$\begin{aligned}
 \partial_t \gamma_0^i &= A^i \circ \gamma_0, & g_0^\rho &= B^\rho \circ \gamma_0, \\
 \gamma_\nu^i &= \sum_\lambda g_\nu^\lambda A_\lambda^i \circ \gamma_0, & \partial_t g_\nu^\rho &= \sum_\lambda g_\nu^\lambda B_\lambda^\rho \circ \gamma_0, \\
 \partial_t \gamma_{\mu\nu}^i &= \sum_j \gamma_{\mu\nu}^j \partial_{y^j} A^i \circ \gamma_0 + \sum_{\eta < \lambda} (g_\mu^\eta g_\nu^\lambda - g_\mu^\lambda g_\nu^\eta) A_{\eta\lambda}^i \circ \gamma_0, \\
 g_{\mu\nu}^\rho &= \sum_j \gamma_{\mu\nu}^j \partial_{y^j} B^\rho \circ \gamma_0 + \sum_{\eta < \lambda} (g_\mu^\eta g_\nu^\lambda - g_\mu^\lambda g_\nu^\eta) B_{\eta\lambda}^\rho \circ \gamma_0,
 \end{aligned}
 \tag{4.2}$$

etc., where use has been made of the fact that for any C^∞ function f on M , $\gamma^\sharp f$ is given by, $f \circ \gamma_0 + \sum_j \gamma_{\mu\nu}^j (\partial_{y^j} f \circ \gamma_0) \theta^\mu \theta^\nu + \dots$. We may now state and prove the following existence and uniqueness theorem in $\mathbf{R}^{1|1}$.

4.1. Theorem. *Let X be a supervector field in $\mathbf{M}$ having level zero at x and a normal form on $\mathbf{R}^{1|1}$. Then there exists a unique (locally defined) morphism Γ , as in (1.1), satisfying (1.2) and (1.3).*

Proof. We may assume $\mathbf{M} \subset \mathbf{R}^{1|1}$. Let $\{y, \theta\}$ be local coordinates defined about x . We shall write $X = (A_0 + A_1\theta)\partial_y + (B_0 + B_1\theta)\partial_\theta$ and Γ as in Theorem 2.1. The system of equations (4.2) reduces to

$$(4.3) \quad \partial_t \gamma_0 = A_0 \circ \gamma_0, \quad g_0 = B_0 \circ \gamma_0, \quad \partial_t g_1 = g_1 B_1 \circ \gamma_0, \quad \gamma_1 = g_1 A_1 \circ \gamma_0.$$

The first equation has a unique solution satisfying $\gamma_0(0, y) = y$. This solution completely determines g_0 by means of the second equation. The third equation has a unique solution satisfying $g_1(0, y) = 1$ (which is the odd part of (1.3)); namely,

$$(4.4) \quad g_1(t, y) = \exp \int_0^t B_1 \circ \gamma_0(s, y) ds.$$

One finally uses this expression to determine γ_1 uniquely via the last equation in (4.3). $\square$

Under the hypotheses of the preceding theorem, the next result gives necessary and sufficient conditions on a supervector field for its integral flow to define a Lie supergroup action of $\mathbf{R}^{1|1}$.

4.2. Theorem. *Let X be a supervector field in $\mathbf{M}$ having level zero at x and a normal form on $\mathbf{R}^{1|1}$. Let $X = X_0 + X_1$ be its decomposition into homogeneous components. The integral flow Γ of X defines a Lie supergroup action of $\mathbf{R}^{1|1}$ on $\mathbf{M}$ if and only if $[X, X] = 0$ and $[X_0, X_1] = 0$.*

Proof. We shall use the same coordinate expressions as in the preceding theorem. In particular,

$$[X, X] = [X_1, X_1] = 2A_1 B_0 \partial_y + 2A_1 B_0' \theta \partial_\theta$$

and

$$[X_0, X_1] = (A_0 A_1' - A_0' A_1 + A_1 B_1) \theta \partial_y + (A_0 B_0' - B_0 B_1) \partial_\theta.$$

Note that (2.5) implies in (4.3) that a first necessary condition is

$$(4.5) \quad A_1(y) B_0(y) = 0, \quad \text{identically.}$$

Let us first assume that $A_1 = 0$ in some neighborhood of x . Then γ_1 is identically zero and, therefore, only the first three equations on the right column of (2.4) need to be checked. Since $g_0 = B_0 \circ \gamma_0$ and γ_0 is an $\mathbf{R}$ -action, the first of the equations is satisfied. Since the third is satisfied when g_1 is defined by means of $g_0(0, y) g_1(t, y) = g_0(t, y)$, only this one needs to be checked, but this holds true only if

$$(4.6) \quad A_0(y) B_0'(y) - B_0(y) B_1(y) = 0,$$

as follows from the explicit form of g_1 in (4.4). This, together with $A_1 = 0$, gives $[X, X] = 0$ and $[X_0, X_1] = 0$.

Let us now assume $B_0 = 0$. This implies $g_0 = 0$, identically. Thus, only the third equation in the right column and the last two in the left in (2.4) need to be checked. Using (4.4) for g_1 , setting $\gamma_1 = g_1 A_1 \circ \gamma_0$, and using the fact that γ_0 is an $\mathbf{R}$ -action, one finds that the third equations in both columns of (2.4) are easily satisfied. What remains is the equation: $\partial_y \gamma_0(t, y) \gamma_1(0, y) = \gamma_1(t, y)$. Since $\partial_y \gamma_0(t, y) = \partial_y \gamma_0(0, y) \exp \int_0^t A_0' \circ \gamma_0(s, y) ds$, one verifies directly that the remaining equation holds true only if

$$(4.7) \quad A_0'(y) A_1(y) - A_0(y) A_1'(y) - A_1(y) B_1(y) = 0.$$

This, together with $B_0 = 0$, again gives $[X, X] = 0$ and $[X_0, X_1] = 0$. $\square$

4.3. Remark. Note that there are nonhomogeneous fields having an $\mathbf{R}^{1|1}$ -action as its flow, e.g.,

$$X = A_0 \partial_y + \left(c e^{\int_0^t B_1(s)/A_0 ds} + B_1 \theta \right) \partial_\theta, \quad c \text{ constant},$$

and

$$X = A_0 \left(1 + \theta c e^{-\int_0^t B_1(s)/A_0 ds} \right) \partial_y + B_1 \theta \partial_\theta, \quad c \text{ constant},$$

that are obtained by solving (4.6) and (4.7), respectively. The only assumption is that $A_0 \neq 0$ so as to yield nontrivial examples.

4.4. Corollary. Let X be a nondegenerate, odd, supervector field on $\mathbf{M}$. Then its integral flow does not define an $\mathbf{R}^{1|1}$ -action on $\mathbf{M}$.

Proof. Such fields have normal form $X = \partial_\theta + \theta \partial_y$ and $[X, X] = 2\partial_y$. $\square$

5. CONCLUDING REMARKS

5.1. Remark. In all cases when $[X_0, X_1] = 0$, the integral flow Γ may be obtained by formal exponentiation of the field as in the C^∞ -theory. In fact, one may define the formal (hybrid) exponential of the superfield X as

$$\exp(tX_0 + \tau X_1) = \sum_{n=0}^{\infty} \frac{1}{n!} (tX_0 + \tau X_1)^n.$$

The lack of ambiguity in computing the integral flow this way follows from the fact that $\exp(tX_0 + \tau X_1) = \exp(tX_0) \exp(\tau X_1) = \exp(\tau X_1) \exp(tX_0)$ if and only if $[X_0, X_1] = 0$.

5.2. *Remark.* A knowledge of the normal forms of level-one (and higher) superfields is important when dealing with an isolated singular point, as some nonelimination of the odd variables may occur in the combination of an even singularity with a level-one field. The simplest example is furnished by $y\partial_y + \lambda\theta\partial_\theta$. There are choices of the C^∞ -function λ for which no change of coordinates can simplify the field to say, $x'\partial_{x'}$. (We are indebted to Professor Sternberg for bringing this point to our attention.) However, the normal form of a level-one field defined in some (m, n) -dimensional superdomain occurs in $\mathbf{R}^{\min(m, n)|n}$, which makes their study awkward. We shall deal with the normal form problem of singular superfields (along the lines of [9, 10, 2]) in a different communication.

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